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PLANE AND SPHERICAL

TRIGONOMETRY.

BY

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PREFACE.

IN preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently, development of functions in series and all other investigations that are important only for the special student have been omitted. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view to awaken a real love for the study. Much time and labor have been spent in devising the simplest proofs for the propositions, and in exhibiting the best methods of arranging the logarithmic work.

The object of the work on Surveying and Navigation is to present these subjects in a clear and intelligible way, according to the best methods in actual use; and also to present them in so small a compass that students in general may find the time to acquire a competent knowledge of these very interesting and important studies.

The author is under particular obligation for assistance to G. A. Hill, A.M., of Cambridge, Mass., and to Prof. James L. Patterson, of Lawrenceville, N.J.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
September, 1882.

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PLANE TRIGONOMETRY.

CHAPTER I.

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES.

§ 1. Definitions.

The sides and angles of a plane triangle are so related that any three given parts, provided at least one of them is a side, determine the shape and the size of the triangle.

Geometry shows how, from three such parts, to *construct* the triangle and find the values of the unknown parts.

Trigonometry shows how to *compute* the unknown parts of a triangle from the numerical values of the given parts.

Geometry shows in a general way that the sides and angles of a triangle are mutually dependent. Trigonometry begins by showing the exact nature of this dependence in the *right triangle*, and for this purpose employs the *ratios of its sides*.

Let MAN (Fig. 1) be an acute angle. If from any points B, D, F,..... in one of its sides perpendiculars BC, DE, FG,..... are let fall to the other side, then the right triangles ABC, ADE, AFG,..... thus formed have the angle A common, and are therefore mutually equiangular and similar. Hence, the ratios of their corresponding sides, pair by pair, are equal. That is,

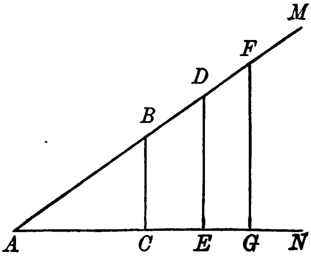

Fig. 1.

$$\frac{AC}{AB}=\frac{AE}{AD}=\frac{AG}{AF}, \quad \frac{AC}{BC}=\frac{AE}{DE}=\frac{AG}{FG}; \text{ etc.}$$

Hence, for every value of an acute angle A there are certain *numbers* that express the values of the ratios of the sides *in all right triangles that have this acute angle A.*

There are altogether six different ratios:

I. The ratio of the opposite leg to the hypotenuse is called the *Sine* of A, and is written $\sin A$.

II. The ratio of the adjacent leg to the hypotenuse is called the *Cosine* of A, and written $\cos A$.

III. The ratio of the opposite leg to the adjacent leg is called the *Tangent* of A, and written $\tan A$.

IV. The ratio of the adjacent leg to the opposite leg is called the *Cotangent* of A, and written $\cot A$.

V. The ratio of the hypotenuse to the adjacent leg is called the *Secant* of A, and written $\sec A$.

VI. The ratio of the hypotenuse to the opposite leg is called the *Cosecant* of A, and written $\csc A$.

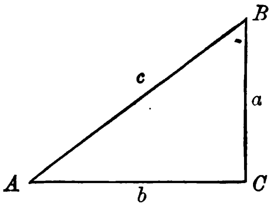

Fig. 2.

In the right triangle ABC (Fig. 2) let a, b, c denote the lengths of the sides opposite to the acute angles A, B, and the right angle C, respectively, these lengths being all expressed in terms of a common unit. Then,

$$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}. \qquad \cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}}.$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}. \qquad \cot A = \frac{b}{a} = \frac{\text{adjacent leg}}{\text{opposite leg}}.$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent leg}}. \qquad \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite leg}}.$$

These six ratios are called the **Trigonometric Functions** of the angle A.

EXERCISE I.

1. What are the functions of the other acute angle B of the triangle ABC (Fig. 2)?

2. Prove that if two angles, A and B, are complements of each other (*i.e.*, if $A+B=90°$), then,

$$\sin A=\cos B,\quad \tan A=\cot B,\quad \sec A=\csc B;$$
$$\cos A=\sin B,\quad \cot A=\tan B,\quad \csc A=\sec B.$$

3. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) 3, 4, 5. (iii.) 8, 15, 17. (v.) 3.9, 8, 8.9.
(ii.) 5, 12, 13. (iv.) 9, 40, 41. (vi.) 1.19, 1.20, 1.69.

4. What condition must be fulfilled by the lengths of the three lines a, b, c (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?

5. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) $2mn,\ m^2-n^2,\ m^2+n^2.$ (iii.) $pqr,\ qrs,\ rsp.$
(ii.) $\dfrac{2xy}{x-y},\ x+y,\ \dfrac{x^2+y^2}{x-y}.$ (iv.) $\dfrac{mn}{pq},\ \dfrac{mv}{sq},\ \dfrac{nr}{ps}.$

6. Prove that the values of a, b, c, in (i.) and (ii.), Example 5, satisfy the condition necessary to make them the sides of a right triangle.

7. What equations of condition must be satisfied by the values of a, b, c, in (iii.) and (iv.), Example 5, in order that the values may represent the sides of a right triangle?

Compute the functions of A and B when,

8. $a=24,\ b=143.$ 11. $a=\sqrt{p^2+q^2},\ b=\sqrt{2pq}.$
9. $a=0.264,\ c=0.265.$ 12. $a=\sqrt{p^2+pq},\ c=p+q.$
10. $b=9.5,\ c=19.3.$ 13. $b=2\sqrt{pq},\ c=p+q.$

Compute the functions of A when,

14. $a = 2b$.
15. $a = \frac{2}{3}c$.
16. $a + b = \frac{5}{4}c$.
17. $a - b = \frac{c}{4}$.
18. Find a if $\sin A = \frac{3}{5}$ and $c = 20.5$.
19. Find b if $\cos A = 0.44$ and $c = 3.5$.
20. Find a if $\tan A = \frac{11}{8}$ and $b = 2\frac{5}{11}$.
21. Find b if $\cot A = 4$ and $a = 17$.
22. Find c if $\sec A = 2$ and $b = 20$.
23. Find c if $\csc A = 6.45$ and $a = 35.6$.

Construct a right triangle; given,

24. $c = 6$, $\tan A = \frac{3}{2}$.
25. $a = 3.5$, $\cos A = \frac{1}{2}$.
26. $b = 2$, $\sin A = 0.6$.
27. $b = 4$, $\csc A = 4$.

28. In a right triangle, $c = 2.5$ miles, $\sin A = 0.6$, $\cos A = 0.8$; compute the legs.

29. Construct (with a protractor) the $\angle$s 20°, 40°, and 70°; determine their functions by measuring the necessary lines, and compare the values obtained in this way with the more correct values given in the following table:

	sin	*cos*	*tan*	*cot*	*sec*	*csc*
20°	0.342	0.940	0.364	2.747	1.064	2.924
40°	0.643	0.766	0.839	1.192	1.305	1.556
70°	0.940	0.342	2.747	0.364	2.924	1.064

30. Find, by means of the above table, the legs of a right triangle if $A = 20°$, $c = 1$; also, if $A = 20°$, $c = 4$.

31. In a right triangle, given $a = 3$ and $c = 5$; find the hypotenuse of a similar triangle in which $a = 240{,}000$ miles.

32. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at the time of observation was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yards?

§ 2. Representation of Functions by Lines.

The functions of an angle, being ratios, are numbers; but we may represent them by *lines* if we first choose a unit of length, and then construct right triangles, such that the denominators of the ratios shall be equal to this unit. The most convenient way to do this is as follows:

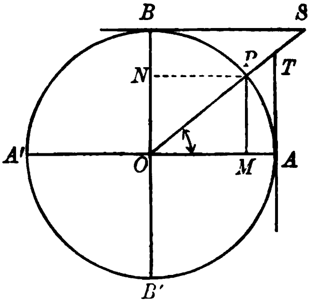

Fig. 3.

About a point O (Fig. 3) as a centre, with a radius equal to one unit of length, describe a circle and draw two diameters AA' and BB' perpendicular to each other.

The circle with radius equal to 1 is called a *unit* circle, AA' the *horizontal*, and BB' the *vertical* diameter.

Let AOP be an acute angle, and let its value (in degrees, etc.) be denoted by x. We may regard the $\angle\ x$ as generated by a radius OP that revolves about O from the position OA to the position shown in the figure; viewed in this way, OP is called the *moving* radius.

Draw $PM \perp$ to OA. In the rt. $\triangle\ OPM$ the hypotenuse $OP = 1$; therefore, $\sin x = PM$; $\cos x = OM$.

Since PM is equal to ON, and ON is the projection of OP on BB', and since OM is the projection of OP on AA', therefore, *in a unit circle*,

$\sin x =$ projection of moving radius on vertical diameter;
$\cos x =$ projection of moving radius on horizontal diameter.

Through A and B draw tangents to the circle meeting OP, produced in T and S, respectively; then, in the rt. $\triangle\ OAT$, the leg $OA = 1$, and in the rt. $\triangle\ OBS$, the leg $OB = 1$; while the $\angle\ OSB = \angle\ x$ (why?). Therefore,

$$\tan x = AT;\ \sec x = OT;\ \cot x = BS;\ \csc x = OS.$$

These six *line* values (as they may be termed) of the functions are all expressed in terms of the radius of the circle as a unit; and it is clear that as the angle varies in value the line values of the functions will always remain equal *numerically* to the ratio values. Hence, in studying the changes in the functions as the angle is supposed to vary, we may employ the simpler line values instead of the ratio values.

Exercise II.

1. Represent by lines the functions of a larger angle than that shown in Fig. 3.

2. Show that $\sin x$ is less than $\tan x$.

3. Show that $\sec x$ is greater than $\tan x$.

4. Show that $\csc x$ is greater than $\cot x$.

Construct the angle x if,

5. $\tan x = 3$.

6. $\csc x = 2$.

7. $\cos x = \frac{1}{4}$.

8. $\sin x = \cos x$.

9. $\sin x = 2\cos x$.

10. $4\sin x = \tan x$.

11. Show that the sine of an angle is equal to one-half the chord of twice the angle.

12. Find x if $\sin x$ is equal to one-half the side of a regular inscribed decagon.

13. Given x and y ($x + y$ being less than 90°); construct the value of $\sin(x+y) - \sin x$.

14. Given x and y ($x + y$ being less than 90°); construct the value of $\tan(x+y) - \sin(x+y) + \tan x - \sin x$.

Given an angle x; construct an angle y such that,

15. $\sin y = 2\sin x$.

16. $\cos y = \frac{1}{2}\cos x$.

17. $\tan y = 3\tan x$.

18. $\sec y = \csc x$.

19. Show by construction that $2\sin A > \sin 2A$.

20. Given two angles A and B ($A + B$ being less than 90°); show that $\sin(A+B) < \sin A + \sin B$.

21. Given $\sin x$ in a unit circle; find the length of a line corresponding in position to $\sin x$ in a circle whose radius is r?

22. In a right triangle, given the hypotenuse c, and also $\sin A = m$, $\cos A = n$; find the legs.

§ 3. CHANGES IN THE FUNCTIONS AS THE ANGLE CHANGES.

If we suppose the ∠ AOP, or x (Fig. 4) to increase gradually by the revolution of the moving radius OP about O, the point P will move along the arc AB towards B, T will move along the tangent AT away from A, S will move along the tangent BS towards B, and M will move along the radius OA towards O.

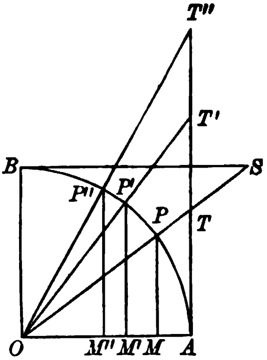

Fig. 4.

Hence, the lines PM, AT, OT will gradually increase in length, and the lines OM, BS, OS, will gradually decrease. That is,

As an acute angle increases, its sine, tangent, and secant also increase, while its cosine, cotangent, and cosecant decrease.

On the other hand, if we suppose x to decrease gradually, the reverse changes in its functions will occur.

If we suppose x to decrease to 0°, OP will coincide with OA and be parallel to BS. Therefore, PM and AT will vanish, OM will become equal to OA, while BS and OS will each be infinitely long, and be represented in value by the symbol ∞.

And if we suppose x to increase to 90°, OP will coincide with OB and be parallel to AT. Therefore, PM and OS will each be equal to OB, OM and BS will vanish, while AT and OT will each be infinite in length.

Hence, as the angle x increases from 0° to 90°,

$\sin x$ increases from 0 to 1,
$\cos x$ decreases from 1 to 0,
$\tan x$ increases from 0 to ∞,
$\cot x$ decreases from ∞ to 0,
$\sec x$ increases from 1 to ∞,
$\csc x$ decreases from ∞ to 1.

The values of the functions of 0° and of 90° are the *limiting* values of the functions of an acute angle. It is evident that (disregarding the limiting values),

Sines and cosines are always less than 1;
Secants and cosecants are always greater than 1;
Tangents and cotangents have all values between 0 and ∞.

Remark. We are now able to understand why the sine, cosine, etc., of an angle are called *functions* of the angle. By a *function* of any magnitude is meant another magnitude which remains the same so long as the first magnitude remains the same, but changes in value for every change in the value of the first magnitude. This, as we now see, is the relation in which the sine, cosine, etc., of an angle stand to the angle.

§ 4. Functions of Complementary Angles.

The general form of two complementary angles is A and $90° - A$.

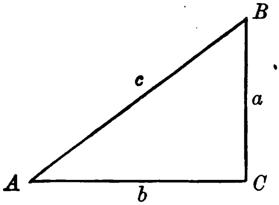

Fig. 5.

In the rt. Δ ABC (Fig. 5)
$A + B = 90°$, hence $B = 90° - A$.
Therefore (§ 1),

$$\begin{aligned} \sin A &= \cos B = \cos(90° - A), \\ \cos A &= \sin B = \sin(90° - A), \\ \tan A &= \cot B = \cot(90° - A), \\ \cot A &= \tan B = \tan(90° - A), \\ \sec A &= \csc B = \csc(90° - A), \\ \csc A &= \sec B = \sec(90° - A). \end{aligned}$$

Therefore,

Each function of an acute angle is equal to the co-named function of the complementary angle.

Note. *Co*sine, *co*tangent, and *co*secant are sometimes called *co-functions;* the words are simply abbreviated forms of *complement's sine, complement's tangent,* and *complement's secant.*

Hence, also,

Any function of an angle between 45° *and* 90° *may be found by taking the co-named function of the complementary angle between* 0° *and* 45°.

EXERCISE III.

1. Express the following functions as functions of the complementary angle:

sin 30°.	tan 89°.	csc 18° 10′.	cot 82° 19′.
cos 45°.	cot 15°.	cos 37° 24′.	csc 54° 46′.

2. Express the following functions as functions of an angle less than 45°:

sin 60°.	tan 57°.	csc 69° 2′.	cot 89° 59′.
cos 75°.	cot 84°.	cos 85° 39′.	csc 45° 1′.

3. Given $\tan 30° = \frac{1}{3}\sqrt{3}$; find $\cot 60°$.
4. Given $\tan A = \cot A$; find A.
5. Given $\cos A = \sin 2A$; find A.
6. Given $\sin A = \cos 2A$; find A.
7. Given $\cos A = \sin(45° - \frac{1}{2}A)$; find A.
8. Given $\cot \frac{1}{2}A = \tan A$; find A.
9. Given $\tan(45° + A) = \cot A$; find A.
10. Find A if $\sin A = \cos 4A$.
11. Find A if $\cot A = \tan 8A$.
12. Find A if $\cot A = \tan nA$.

§ 5. RELATIONS OF THE FUNCTIONS OF AN ANGLE.

Since (Fig. 5) $a^2 + b^2 = c^2$, therefore,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \quad \text{or} \quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Therefore (§ 1), $(\sin A)^2 + (\cos A)^2 = 1$;

or, as usually written for convenience,

$$\sin^2 A + \cos^2 A = 1. \qquad [1]$$

That is: *The sum of the squares of the sine and the cosine of an angle is equal to unity.*

Formula [1] enables us to find the cosine of an angle when the sine is known, and *vice versa.* The values of sin A and of cos A deduced from [1] are:

$$\sin A = \sqrt{1-\cos^2 A}, \quad \cos A = \sqrt{1-\sin^2 A}.$$

Since $$\frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b},$$

therefore (§ 1), $$\mathbf{\tan A = \frac{\sin A}{\cos A}}. \qquad [2]$$

That is: *The tangent of an angle is equal to the sine divided by the cosine.*

Formula [2] enables us to find the tangent of an angle when the sine and the cosine are known.

Since $$\frac{a}{c} \times \frac{c}{a} = 1, \quad \frac{b}{c} \times \frac{c}{b} = 1, \text{ and } \frac{a}{b} \times \frac{b}{a} = 1,$$

therefore (§ 1), $$\left.\begin{aligned} \mathbf{\sin A \times \csc A = 1} \\ \mathbf{\cos A \times \sec A = 1} \\ \mathbf{\tan A \times \cot A = 1} \end{aligned}\right\}. \qquad [3]$$

That is: *The sine and the cosecant of an angle, the cosine and secant, and the tangent and cotangent, pair by pair, are reciprocals.*

The equations in [3] enable us to find an unknown function contained in any pair of these reciprocals when the other function in this pair is known.

Exercise IV.

1. Prove Formulas [1] – [3], using for the functions the line values in unit circle given in § 3.

2. Prove that $1 + \tan^2 A = \sec^2 A$.

   Hint. Divide the terms of the equation $a^2 + b^2 = c^2$ by b^2.

3. Prove that $1 + \cot^2 A = \csc^2 A$.

4. Prove that $\cot A = \frac{\cos A}{\sin A}$.

§ 6. APPLICATION OF FORMULAS [1]–[3].

Formulas [1], [2], and [3] enable us, when any one function of an angle is given, to find all the others. A given value of any one function, therefore, determines all the others.

EXAMPLE 1. Given $\sin A = \frac{2}{3}$; find the other functions.

By [1], $\cos A = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{1}{3}\sqrt{5}.$

By [2], $\tan A = \dfrac{2}{3} \div \dfrac{1}{3}\sqrt{5} = \dfrac{2}{3} \times \dfrac{3}{\sqrt{5}} = \dfrac{2}{\sqrt{5}}.$

By [3], $\cot A = \dfrac{\sqrt{5}}{2}, \quad \sec A = \dfrac{3}{\sqrt{5}}, \quad \csc A = \dfrac{3}{2}.$

EXAMPLE 2. Given $\tan A = 3$; find the other functions.

By [2], $\dfrac{\sin A}{\cos A} = 3.$

And by [1], $\sin^2 A + \cos^2 A = 1.$

If we solve these equations (regarding $\sin A$ and $\cos A$ as two unknown quantities), we find that,

$$\sin A = 3\sqrt{\tfrac{1}{10}}, \quad \cos A = \sqrt{\tfrac{1}{10}}.$$

Then by [3], $\cot A = \frac{1}{3}, \quad \sec A = \sqrt{10}, \quad \csc A = \frac{1}{3}\sqrt{10}.$

EXAMPLE 3. Given $\sec A = m$; find the other functions.

By [3], $\cos A = \dfrac{1}{m}.$

By [1], $\sin A = \sqrt{1 - \dfrac{1}{m^2}} = \sqrt{\dfrac{m^2 - 1}{m^2}} = \dfrac{1}{m}\sqrt{m^2 - 1}.$

By [2], [3], $\tan A = \sqrt{m^2 - 1}, \quad \cot A = \dfrac{1}{\sqrt{m^2 - 1}},$

$$\csc A = \frac{m}{\sqrt{m^2 - 1}}.$$

EXERCISE V.

Find the values of the other functions when,

1. $\sin A = \frac{12}{13}$.
2. $\sin A = 0.8$.
3. $\cos A = \frac{60}{61}$.
4. $\cos A = 0.28$.
5. $\tan A = \frac{4}{3}$.
6. $\cot A = 1$.
7. $\cot A = 0.5$.
8. $\sec A = 2$.
9. $\csc A = \sqrt{2}$.
10. $\sin A = m$.
11. $\sin A = \dfrac{2m}{1 + m^2}$.
12. $\cos A = \dfrac{2mn}{m^2 + n^2}$.

13. Given $\tan 45° = 1$; find the other functions of 45°.

14. Given $\sin 30° = \frac{1}{2}$; find the other functions of 30°.

15. Given $\csc 60° = \frac{2}{3}\sqrt{3}$; find the other functions of 60°.

16. Given $\tan 15° = 2 - \sqrt{3}$; find the other functions of 15°.

17. Given $\cot 22° 30' = \sqrt{2} + 1$; find the other functions of 22° 30′.

18. Given $\sin 0° = 0$; find the other functions of 0°.

19. Given $\sin 90° = 1$; find the other functions of 90°.

20. Given $\tan 90° = \infty$; find the other functions of 90°.

21. Express the values of all the other functions in terms of $\sin A$.

22. Express the values of all the other functions in terms of $\cos A$.

23. Express the values of all the other functions in terms of $\tan A$.

24. Express the values of all the other functions in terms of $\cot A$.

25. Given $2 \sin A = \cos A$; find $\sin A$ and $\cos A$.

26. Given $4 \sin A = \tan A$; find $\sin A$ and $\tan A$.

27. If $\sin A : \cos A = 9 : 40$, find $\sin A$ and $\cos A$,

28. Transform the quantity $\tan^2 A + \cot^2 A - \sin^2 A - \cos^2 A$ into a form containing only $\cos A$.

29. Prove that $\sin A + \cos A = (1 + \tan A) \cos A$.

30. Prove that $\tan A + \cot A = \sec A \times \csc A$.

§ 7. Functions of 45°.

Let ABC (Fig. 6) be an isosceles right triangle, in which the length of the hypotenuse AB is equal to 1; then AC is equal to BC, and the angle A is equal to 45°. Since $\overline{AC}^2 + \overline{BC}^2 = 1$, therefore $2\,\overline{AC}^2 = 1$, and $AC = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}$.

Therefore (§ 1),

$$\sin 45° = \cos 45° = \tfrac{1}{2}\sqrt{2}.$$
$$\tan 45° = \cot 45° = 1.$$
$$\sec 45° = \csc 45° = \sqrt{2}.$$

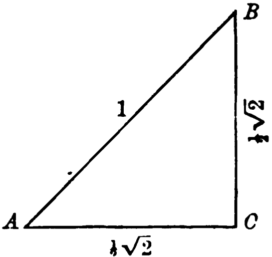

Fig. 6.

§ 8. Functions of 30° and 60°.

Let ABC be an equilateral triangle in which the length of each side is equal to 1; and let CD bisect the angle C. Then CD is perpendicular to AB and bisects AB; hence, $AD = \tfrac{1}{2}$, and $CD = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}$.

In the right triangle ADC, the angle $ACD = 30°$, and the angle $CAD = 60°$.

Whence (§ 1),

$$\sin 30° = \cos 60° = \tfrac{1}{2}.$$
$$\cos 30° = \sin 60° = \tfrac{1}{2}\sqrt{3}.$$
$$\tan 30° = \cot 60° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$
$$\cot 30° = \tan 60° = \sqrt{3}.$$
$$\sec 30° = \csc 60° = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$
$$\csc 30° = \sec 60° = 2.$$

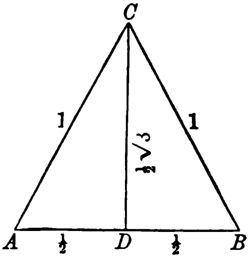

Fig. 7.

The results for sine and cosine of 30°, 45°, and 60° may be easily remembered by arranging them in the following form:

Angle . . .	30°	45°	60°	$\tfrac{1}{2} = 0.5$
Sine	$\tfrac{1}{2}$	$\tfrac{1}{2}\sqrt{2}$	$\tfrac{1}{2}\sqrt{3}$	$\tfrac{1}{2}\sqrt{2} = 0.70711$
Cosine . . .	$\tfrac{1}{2}\sqrt{3}$	$\tfrac{1}{2}\sqrt{2}$	$\tfrac{1}{2}$	$\tfrac{1}{2}\sqrt{3} = 0.86603$

CHAPTER II.

THE RIGHT TRIANGLE.

§ 9. The Given Parts.

In order to solve a right triangle, two parts besides the right angle must be given, one of them at least being a side.

The two given parts may be :

I. An acute angle and the hypotenuse.
II. An acute angle and the opposite leg.
III. An acute angle and the adjacent leg.
IV. The hypotenuse and a leg.
V. The two legs.

§ 10. Case I.

Given $A = 34^\circ 28'$ and $c = 18.75$; required B, a, b.

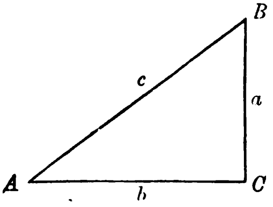

Fig. 8.

1. $B = 90^\circ - A = 55^\circ 32'$.

2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.

3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

$$\begin{aligned}
\log a &= \log c + \log \sin A \\
\log c &= 1.27300 \\
\log \sin A &= \underline{9.75276}\,* \\
\log a &= 1.02576 \\
a &= 10.61
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\log b &= \log c + \log \cos A \\
\log c &= 1.27300 \\
\log \cos A &= \underline{9.91617} \\
\log b &= 1.18917 \\
b &= 15.459
\end{aligned}$$

---

* For Logarithms, and directions how to use them, see Wentworth and Hill's *Five-place Tables.*

When -10 belongs to a logarithm or cologarithm, and is not written, it must be *remembered* that the logarithm or cologarithm is 10 too large.

§ 11. Case II.

Given $A = 62° 10'$, $a = 78$; find B, b, c.

1. $B = 90° - A = 27° 50'$.
2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.
3. $\frac{a}{c} = \sin A$.

$\therefore\ a = c \sin A$, and $c = \frac{a}{\sin A}$.

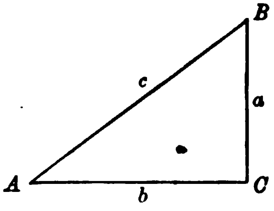

Fig. 9.

$$\begin{aligned} \log b &= \log a + \log\cot A \\ \log a &= 1.89209 \\ \log\cot A &= \underline{9.72262} \\ \log b &= 1.61471 \\ b &= 41.182 \end{aligned}$$

$$\begin{aligned} \log c &= \log a + \operatorname{colog}\sin A \\ \log a &= 1.89209 \\ \operatorname{colog}\sin A &= \underline{0.05340} \\ \log c &= 1.94549 \\ c &= 88.204 \end{aligned}$$

§ 12. Case III.

Given $A = 50° 2'$, $b = 88$; find B, a, c.

1. $B = 90° - A = 39° 58'$.
2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.
3. $\frac{b}{c} = \cos A$.

$\therefore\ b = c \cos A$, and $c = \frac{b}{\cos A}$.

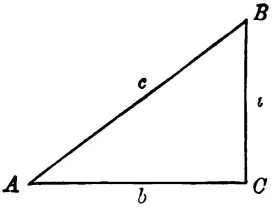

Fig. 10.

$$\begin{aligned} \log a &= \log b + \log\tan A \\ \log b &= 1.94448 \\ \log\tan A &= \underline{10.0767} \\ \log a &= 2.02118 \\ a &= 105.0 \end{aligned}$$

$$\begin{aligned} \log c &= \log b + \operatorname{colog}\cos A \\ \log b &= 1.94448 \\ \operatorname{colog}\cos A &= \underline{0.19223} \\ \log c &= 2.13671 \\ c &= 137.0 \end{aligned}$$

§ 13. Case IV.

Given $c = 58.40$, $a = 47.55$; find A, B, b.

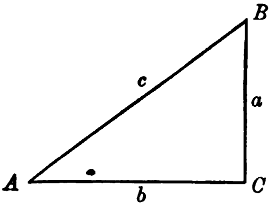

Fig. 11.

1. $\sin A = \frac{a}{c}$.
2. $B = 90° - A$.
3. $\frac{b}{a} = \cot A$; $\quad \therefore b = a \cot A$.

$\log \sin A = \log a + \operatorname{colog} c$
$\log a = 1.67715$
$\operatorname{colog} c = 8.23359$
$\log \sin A = 9.91074$
$A = 54° 31'$
$B = 35° 29'$

$\log b = \log a + \log \cot A$
$\log a = 1.67715$
$\log \cot A = 9.85300$
$\log b = 1.53015$
$b = 33.896$

§ 14. Case V.

Given $a = 40$, $b = 27$; find A, B, c.

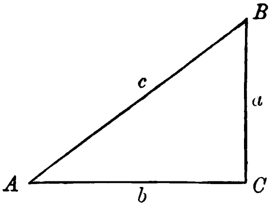

Fig. 12.

1. $\tan A = \frac{a}{b}$.
2. $B = 90° - A$.
3. $\frac{a}{c} = \sin A$.

$\therefore a = c \sin A$; $\quad \therefore c = \frac{a}{\sin A}$.

$\log \tan A = \log a + \operatorname{colog} b$
$\log a = 1.60206$
$\operatorname{colog} b = 8.56864$
$\log \tan A = 10.17070$
$A = 55° 59'$
$B = 34° 1'$

$\log c = \log a + \operatorname{colog} \sin A$
$\log a = 1.60206$
$\operatorname{colog} \sin A = 0.08152$
$\log c = 1.68358$
$c = 48.259$

§ 15. General Method of Solving the Right Triangle.

From these five cases it appears that the general method of finding an unknown part in a right triangle is as follows:

Choose from the equation $A + B = 90°$, *and the equations that define the functions of the angles, an equation in which the required part only is unknown; solve this equation, if necessary, to find the value of the unknown part; then compute the value, using logarithms whenever convenient.*

Note 1. In Cases IV. and V. the unknown side may also be found by Geometry, from the equation $a^2 + b^2 = c^2$; whence we obtain

(for Case IV.) $b = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}$;

(for Case V.) $c = \sqrt{a^2 + b^2}$.

These equations express the values of b and c directly in terms of the two given sides; and if the values of the sides are simple numbers (*e.g.* 5, 12, 13), it is often easier to find b or c in this way. But this value of c is not adapted to logarithms, and this value of b is not so readily worked out by logarithms as the value of b given in § 13.

Note 2. In Case IV. if the given sides (here a and c) are nearly alike in value, then A is near 90°, and its value cannot be accurately found from the tables, because the sines of large angles differ little in value (as is evident from Fig. 4). In this case it is better to find B first, by means of a formula proved later (see page 47); viz.,

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c-a}{c+a}};$$

and to find b by the method given in Note 1, since the same logarithms are used in both cases.

Example. Given $a = 49$, $c = 50$; find A, B, b.

$$\begin{aligned}
\log b &= \tfrac{1}{2}[\log(c-a) + \log(c+a)]\\
c - a &= 1\\
c + a &= 99\\
\log(c-a) &= 0.00000\\
\log(c+a) &= \underline{1.99564}\\
&\ 2\overline{)1.99564}\\
\log b &= 0.99782\\
b &= 9.95
\end{aligned}$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}B &= \tfrac{1}{2}[\log(c-a)\\
&\qquad + \operatorname{colog}(c+a)]\\
\log(c-a) &= 0.00000\\
\operatorname{colog}(c+a) &= \underline{8.00436}\\
&\ 2\overline{)8.00436}\\
\log\tan\tfrac{1}{2}B &= 9.00218\\
\tfrac{1}{2}B &= 5°\,44'\,21''\\
B &= 11°\,29'\\
A &= 78°\,31'
\end{aligned}$$

§ 16. AREA OF THE RIGHT TRIANGLE.

It is shown in Geometry that the area of a triangle is equal to one-half the product of the base by the altitude.

Therefore, if a and b denote the legs of a right triangle, and F the area,

$$F = \tfrac{1}{2}ab.$$

By means of this formula the area may always be found when a and b are given or have been computed.

For example: Find the area, having given:

CASE I. (§ 10).

$A = 34° 28'$, $c = 18.75$.

First find (as in § 10) $\log a$ and $\log b$.

$$\log(F) = \log a + \log b + \operatorname{colog} 2$$

$$\begin{aligned} \log a &= 1.02578 \\ \log b &= 1.18915 \\ \operatorname{colog} 2 &= \underline{9.69897} \\ \log(F) &= 1.91390 \\ F &= 82.016 \end{aligned}$$

CASE IV. (§ 13).

$a = 47.54$, $c = 58.40$.

First find (as in § 13) $\log a$ and $\log b$.

$$\log(F) = \log a + \log b + \operatorname{colog} 2$$

$$\begin{aligned} \log a &= 1.67715 \\ \log b &= 1.53025 \\ \operatorname{colog} 2 &= \underline{9.69897} \\ \log(F) &= 2.90637 \\ F &= 806.06 \end{aligned}$$

EXERCISE VI.

1. In Case II. give another way of finding c, after b has been found.
2. In Case III. give another way of finding c, after a has been found.
3. In Case IV. give another way of finding b, after the angles have been found.
4. In Case V. give another way of finding c, after the angles have been found.
5. Given B and c; find A, a, b.
6. Given B and b; find A, a, c.
7. Given B and a; find A, b, c.
8. Given b and c; find A, B, a.

Solve the following triangles:

	GIVEN:		REQUIRED:		
9	$a = 6$,	$c = 12$.	$A = 30^\circ$,	$B = 60^\circ$,	$b = 10.392$.
10	$A = 60^\circ$,	$b = 4$.	$B = 30^\circ$,	$c = 8$,	$a = 6.9282$.
11	$A = 30^\circ$,	$a = 3$.	$B = 60^\circ$,	$c = 6$,	$b = 5.1961$.
12	$a = 4$,	$b = 4$.	$A = B = 45^\circ$, $c = 5.6568$.		
13	$a = 2$,	$c = 2.82843$.	$A = B = 45^\circ$, $b = 2$.		
14	$c = 627$,	$A = 23^\circ\ 30'$.	$B = 66^\circ\ 30'$,	$a = 250.02$,	$b = 575.0$.
15	$c = 2280$,	$A = 28^\circ\ 5'$.	$B = 61^\circ\ 55'$,	$a = 1073.3$,	$b = 2011.6$.
16	$c = 72.15$,	$A = 39^\circ\ 34'$.	$B = 50^\circ\ 26'$,	$a = 45.958$,	$b = 55.620$.
17	$c = 1$,	$A = 36^\circ$.	$B = 54^\circ$,	$a = 0.58779$,	$b = 0.80902$.
18	$c = 200$,	$B = 21^\circ\ 47'$.	$A = 68^\circ\ 13'$,	$a = 185.73$,	$b = 74.22$.
19	$c = 93.4$,	$B = 76^\circ\ 25'$.	$A = 13^\circ\ 35'$,	$a = 21.936$,	$b = 90.788$.
20	$a = 637$,	$A = 4^\circ\ 35'$.	$B = 85^\circ\ 25'$,	$b = 7946$,	$c = 7971.5$.
21	$a = 48.532$,	$A = 36^\circ\ 44'$.	$B = 53^\circ\ 16'$,	$b = 65.031$,	$c = 81.144$.
22	$a = 0.0008$,	$A = 86^\circ$.	$B = 4^\circ$,	$b = 0.0000559$,	$c = 0.000802$
23	$b = 50.937$,	$B = 43^\circ\ 48'$.	$A = 46^\circ\ 12'$,	$a = 53.116$,	$c = 73.59$.
24	$b = 2$,	$B = 3^\circ\ 38'$.	$A = 86^\circ\ 22'$,	$a = 31.497$,	$c = 31.560$.
25	$a = 992$,	$B = 76^\circ\ 19'$.	$A = 13^\circ\ 41'$,	$b = 4074.5$.	$c = 4193.6$.
26	$a = 73$,	$B = 68^\circ\ 52'$.	$A = 21^\circ\ 8$,	$b = 188.86$,	$c = 202.47$.
27	$a = 2.189$,	$B = 45^\circ\ 25'$.	$A = 44^\circ\ 35'$,	$b = 2.2211$,	$c = 3.1185$.
28	$b = 4$,	$A = 37^\circ\ 56'$.	$B = 52^\circ\ 4'$,	$a = 3.1176$,	$c = 5.0714$.
29	$c = 8590$,	$a = 4476$.	$A = 31^\circ\ 24'$,	$B = 58^\circ\ 36'$,	$b = 7332.8$.
30	$c = 86.53$,	$a = 71.78$.	$A = 56^\circ\ 3'$,	$B = 33^\circ\ 57'$,	$b = 48.324$.
31	$c = 9.35$,	$a = 8.49$.	$A = 65^\circ\ 14'$,	$B = 24^\circ\ 46'$,	$b = 3.917$.
32	$c = 2194$,	$b = 1312.7$.	$A = 53^\circ\ 15'$,	$B = 36^\circ\ 45'$,	$a = 1758$.
33	$c = 30.69$,	$b = 18.256$.	$A = 53^\circ\ 30'$,	$B = 36^\circ\ 30'$,	$a = 24.67$.
34	$a = 38.313$,	$b = 19.522$.	$A = 63^\circ$,	$B = 27^\circ$,	$c = 43$.
35	$a = 1.2291$,	$b = 14.950$.	$A = 4^\circ\ 42'$,	$B = 85^\circ\ 18'$,	$c = 15$.
36	$a = 415.38$,	$b = 62.080$.	$A = 81^\circ\ 30'$,	$B = 8^\circ\ 30'$,	$c = 420$.
37	$a = 13.690$,	$b = 16.926$.	$A = 38^\circ\ 58'$,	$B = 51^\circ\ 2'$,	$c = 21.769$.
38	$c = 91.92$,	$a = 2.19$.	$A = 1^\circ 21' 55''$,	$B = 88^\circ 38' 5''$,	$b = 91.894$.

Compute the unknown parts and also the area, having given:

39. $a = 5$, $b = 6$.
40. $a = 0.615$, $c = 70$.
41. $b = \sqrt[3]{2}$, $c = \sqrt{3}$.
42. $a = 7$, $A = 18^\circ\ 14'$.
43. $b = 12$, $A = 29^\circ\ 8'$.
44. $c = 68$, $A = 69^\circ\ 54'$.
45. $c = 27$, $B = 44^\circ\ 4'$.
46. $a = 47$, $B = 48^\circ\ 49'$.
47. $b = 9$, $B = 34^\circ\ 44'$.
48. $c = 8.462$, $B = 86^\circ\ 4'$.

49. Find the value of F in terms of c and A.

50. Find the value of F in terms of a and A.

51. Find the value of F in terms of b and A.

52. Find the value of F in terms of a and c.

53. Given $F = 58$, $a = 10$; solve the triangle.

54. Given $F = 18$, $b = 5$; solve the triangle.

55. Given $F = 12$, $A = 29^\circ$; solve the triangle.

56. Given $F = 100$, $c = 22$; solve the triangle.

57. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

58. Find the legs of a right triangle if the hypotenuse $= 6$, and one angle is twice the other.

59. In a right triangle given c, and $A = nB$; find a and b.

60. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs; find the angles.

The angle of elevation of an object (or angle of depression, if the object is below the level of the observer) is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.

61. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be $60^\circ 30'$; find the height of the steeple.

62. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24°; find the distance of the boat from the foot of the rock.

63. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top $3^\circ 30'$?

64. In order to find the breadth of a river a distance AB was measured along the bank, the point A being directly opposite a tree C on the other side. The angle ABC was also measured. If $AB = 96$ feet, and $ABC = 21^\circ 14'$, find the breadth of the river.

If $ABC = 45^\circ$, what would be the breadth of the river?

65. Find the angle of elevation of the sun when a tower a feet high casts a horizontal shadow b feet long. Find the angle when $a = 120$, $b = 70$.

66. How high is a tree that casts a horizontal shadow b feet in length when the angle of elevation of the sun is A°? Find the height of the tree when $b = 80$, $A = 50^\circ$.

67. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

68. A ship is sailing due north-east with a velocity of 10 miles an hour. Find the rate at which she is moving due north, and also due east.

69. In front of a window 20 feet high is a flower-bed 6 feet wide. How long must a ladder be to reach from the edge of the bed to the window?

70. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

71. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15°. Find the height of the hill.

72. A fort stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30°, and at a point 100 feet nearer the fort it is 45°. How high is the fort?

73. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of the steeple were found to be 40° and 51° respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the distance from the belfry to the top of the steeple.

74. The angle of elevation of the top of an inaccessible fort C, observed from a point A, is 12°. At a point B, 219 feet from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.

§ 17. The Isosceles Triangle.

An isosceles triangle is divided by the perpendicular from the vertex to the base into two equal right triangles.

Therefore, an isosceles triangle is determined by any two parts that determine one of these right triangles.

Let the parts of an isosceles triangle ABC (Fig. 13), among which the altitude CD is to be included, be denoted as follows:

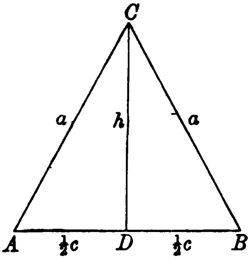

Fig. 13.

$a =$ one of the equal sides.
$c =$ the base.
$h =$ the altitude.
$A =$ one of the equal angles.
$C =$ the angle at the vertex.

For example: Given a and c; required A, C, h.

1. $\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}$.
2. $C + 2A = 180°$; $\therefore C = 180° - 2A = 2(90° - A)$.
3. h may be found directly in terms of a and c from the equation

$$h^2 + \frac{c^2}{4} = a^2,$$

which gives $$h = \sqrt{(a - \tfrac{1}{2}c)(a + \tfrac{1}{2}c)}.$$

But it is better to find the angles first, and then find h from either one of the two equations,

$$\frac{h}{a} = \sin A, \quad \text{or} \quad \frac{h}{\frac{1}{2}c} = \tan A.$$

whence $$h = a \sin A, \text{ or } \quad h = \tfrac{1}{2}c \tan A.$$

The numerical values of A, C, and h may be computed by the aid of logarithms, as in the case of the right triangle.

The area F of the triangle may be found, when c and h are given or have been found, by means of the formula

$$F = \tfrac{1}{2}ch.$$

EXERCISE VII.

In an isosceles triangle:

1. Given a and A; find C, c, h.
2. Given a and C; find A, c, h.
3. Given c and A; find C, a, h.
4. Given c and C; find A, a, h.
5. Given h and A; find C, a, c.
6. Given h and C; find A, a, c.
7. Given a and h; find A, C, c.
8. Given c and h; find A, C, a.
9. Given $a = 14.3$, $c = 11$; find A, C, h.
10. Given $a = 0.295$, $A = 68° 10'$; find c, h, F.
11. Given $c = 2.352$, $C = 69° 49'$; find a, h, F.
12. Given $h = 7.4847$, $A = 76° 14'$; find a, c, F.
13. Given $a = 6.71$, $h = 6.60$; find A, C, c.
14. Given $c = 9$, $h = 20$; find A, C, a.
15. Given $c = 147$, $F = 2572.5$; find A, C, a, h.
16. Given $h = 16.8$, $F = 43.68$; find A, C, a, c.
17. Find the value of F in terms of a and c.
18. Find the value of F in terms of a and C.
19. Find the value of F in terms of a and A.
20. Find the value of F in terms of h and C.
21. A barn is 40 × 80 feet, the pitch of the roof is 45°; find the length of the rafters and the area of both sides of the roof.
22. In a unit circle what is the length of the chord corresponding to the angle 45° at the centre?
23. If the radius of a circle $= 30$, and the length of a chord $= 44$, find the angle at the centre.
24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133°.
25. What is the angle at the centre of a circle if the corresponding chord is equal to $\frac{2}{3}$ of the radius?
26. Find the area of a circular sector if the radius of the circle $= 12$, and the angle of the sector $= 30°$.

§ 18. THE REGULAR POLYGON.

Lines drawn from the centre of a regular polygon (Fig. 14) to the vertices are radii of the circumscribed circle; and lines drawn from the centre to the middle points of the sides are radii of the inscribed circle. These lines divide the polygon into equal right triangles. Therefore, a regular polygon is determined by a right triangle whose sides are the radius of the circumscribed circle, the radius of the inscribed circle, and half of one side of the polygon.

If the polygon has n sides, the angle of this right triangle at the centre is equal to

$$\frac{1}{2}\left(\frac{360°}{n}\right) \text{ or } \frac{180°}{n}.$$

If, also, a side of the polygon, or one of the above-mentioned radii, is given, this triangle may be solved, and the solution gives the unknown parts of the polygon.

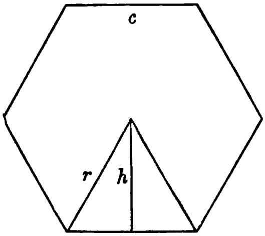

Fig. 14.

Let,

n = number of sides.
c = length of one side.
r = radius of circumscribed circle.
h = radius of inscribed circle.
p = the perimeter.
F = the area.

Then, by Geometry,

$$F = \tfrac{1}{2} hp.$$

EXERCISE VIII.

1. Given $n = 10$, $c = 1$; find r, h, F.
2. Given $n = 12$, $p = 70$; find r, h, F.
3. Given $n = 18$, $r = 1$; find h, p, F.
4. Given $n = 20$, $r = 20$; find h, c, F.
5. Given $n = 8$, $h = 1$; find r, c, F.
6. Given $n = 11$, $F = 20$; find r, h, c.
7. Given $n = 7$, $F = 7$; find r, h, p.

8. Find the side of a regular decagon inscribed in a unit circle.

9. Find the side of a regular decagon circumscribed about a unit circle.

10. If the side of an inscribed regular hexagon is equal to 1, find the side of an inscribed regular dodecagon.

11. Given n and c, and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c.

12. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.

13. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

14. From a square whose side is equal to 1 the corners are cut away so that a regular octagon is left. Find the area of this octagon.

15. Find the area of a regular pentagon if its diagonals are each equal to 12.

16. The area of an inscribed regular pentagon is 331.8; find the area of a regular polygon of 11 sides inscribed in the same circle.

17. The perimeter of an equilateral triangle is 20; find the area of the inscribed circle.

18. The area of a regular polygon of 16 sides, inscribed in a circle, is 100; find the area of a regular polygon of 15 sides, inscribed in the same circle.

19. A regular dodecagon is circumscribed about a circle, the circumference of which is equal to 1; find the perimeter of the dodecagon.

20. The area of a regular polygon of 25 sides is equal to 40; find the area of the ring comprised between the circumferences of the inscribed and the circumscribed circles.

CHAPTER III.

GONIOMETRY.

§ 21. DEFINITION OF GONIOMETRY.

In order to prepare the way for the solution of an oblique triangle, we now proceed to extend the definitions of the trigonometric functions to angles of all magnitudes, and to deduce certain useful relations of the functions of different angles.

That branch of Trigonometry which treats of trigonometric functions in general, and of their relations, is called **Goniometry.**

§ 22. ANGLES OF ANY MAGNITUDE.

Let the radius *OP* of a circle (Fig. 16) generate an angle by turning about the centre *O*. This angle will be measured by the arc described by the point *P*; and it may have any magnitude, because the arc described by *P* may have any magnitude.

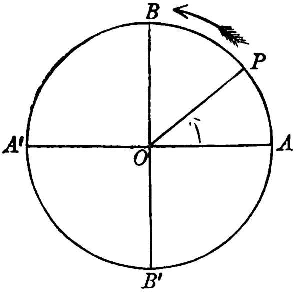

Fig. 16.

Let the horizontal line *OA* be the initial position of *OP*, and let *OP* revolve in the direction shown by the arrow, or opposite to the way clock-hands revolve. Let, also, the four quadrants into which the circle is divided by the horizontal and vertical diameters *AA'*, *BB'*, be numbered I., II., III., IV., in the direction of the motion.

During one revolution OP will form with OA all angles from 0° to 360°. Any particular angle is said to be an angle of the quadrant in which OP lies; so that,

Angles between 0° and 90° are angles of Quadrant I.
Angles between 90° and 180° are angles of Quadrant II.
Angles between 180° and 270° are angles of Quadrant III.
Angles between 270° and 360° are angles of Quadrant IV.

If OP make another revolution, it will describe all angles from 360° to 720°, and so on.

If OP, instead of making another revolution in the direction of the arrow, be supposed to revolve *backwards* about O, this backward motion tends to undo or cancel the original forward motion. Hence, the angle thus generated must be regarded as a *negative* angle; and this negative angle may obviously have any magnitude. Thus we arrive at the conception of an angle of any magnitude, positive or negative.

§ 23. GENERAL DEFINITIONS OF THE FUNCTIONS.

The definitions of the trigonometric functions may be extended to all angles, by making the functions of any angle equal to the line values in a unit circle drawn for the angle in question, as explained in § 3. *But the lines that represent the sine, cosine, tangent, and cotangent must be regarded as negative, if they are opposite in direction to the lines that represent the corresponding functions of an angle in the first quadrant; and the lines that represent the secant and cosecant must be regarded as negative, if they are opposite in direction to the moving radius.*

Figs. 17–20 show the functions drawn for an angle AOP in each quadrant taken in order. In constructing them, it must be remembered that the tangents to the circle are *always* drawn through A and B, *never* through A' or B'.

Let the angle AOP be denoted by x; then, in each figure the *absolute* values of the functions, that is, their values without regard to the signs + and –, are as follows:

$\sin x = MP,\qquad \tan x = AT,\qquad \sec x = OT.$

$\cos x = OM,\qquad \cot x = BS,\qquad \csc x = OS.$

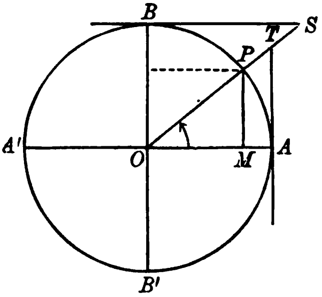

Fig. 17.

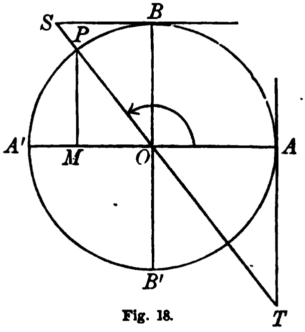

Fig. 18.

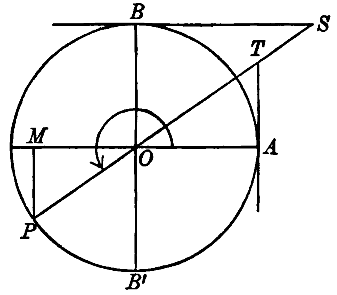

Fig. 19.

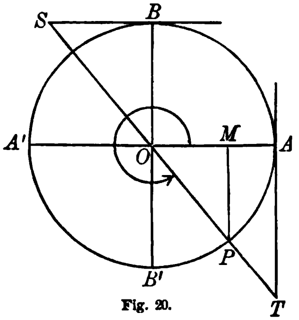

Fig. 20.

Keeping in mind the position of the points A and B, we may define in words the first four functions of the angle x thus:

$\sin x =$ the vertical projection of the moving radius;

$\cos x =$ the horizontal projection of the moving radius;

$\tan x = \Big\{$ the distance measured along a tangent to the circle from the beginning of the first quadrant to the moving radius produced;

$\cot x = \Big\{$ the distance measured along a tangent to the circle from the end of the first quadrant to the moving radius produced.

Sec x and csc x are the distances from the centre of the circle measured along the moving radius produced to the tangent and cotangent respectively.

§ 24. Algebraic Signs of the Functions.

The lengths of the lines, defined above as the functions of any angle, are expressed numerically in terms of the radius of the circle as the unit. But, before these lengths can be treated as algebraic quantities, they must have the sign + or − prefixed, according to the condition stated in § 23.

The reason for this condition lies in that fundamental relation between algebraic and geometric magnitudes, in virtue of which *contrary signs in Algebra correspond to opposite directions in Geometry.*

The sine MP and the tangent AT always extend from the horizontal diameter, but sometimes *upwards* and sometimes *downwards;* the cosine OM and the cotangent BS always extend from the vertical diameter, but sometimes *towards the right* and sometimes *towards the left.* The functions of an angle in the first quadrant are assumed to be positive. Therefore,

1. Sines and tangents extending from the horizontal diameter *upwards*, are positive; *downwards*, negative.
2. Cosines and cotangents extending from the vertical diameter *towards the right*, are positive; *towards the left*, are negative.

The signs of the secant and cosecant are always made to agree with those of the cosine and sine respectively. This agreement is secured if secants and cosecants extending from the centre, *in the direction of the moving radius*, are considered positive; *in the opposite direction*, negative.

Hence, the signs of the functions for each quadrant are:

	I.	II.	III.	IV.
Sine and cosecant	+	+	−	−
Cosine and secant	+	−	−	+
Tangent and cotangent	+	−	+	−

§ 25. FUNCTIONS OF A VARIABLE ANGLE.

Let the angle x increase continuously from 0° to 360°; what changes will the values of its functions undergo?

It is easy, by reference to Figs. 21–24, to trace these changes throughout all the quadrants.

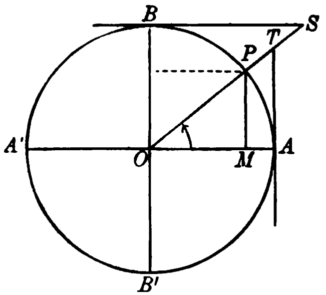

Fig. 21.

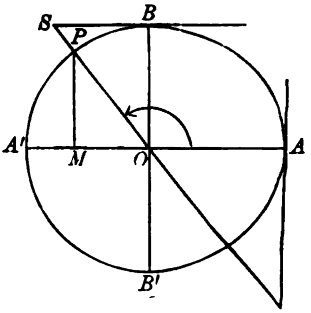

Fig. 22.

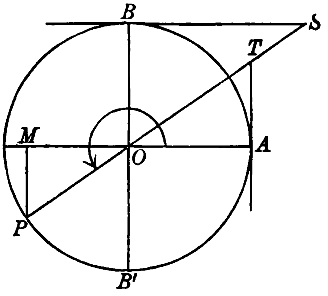

Fig. 23.

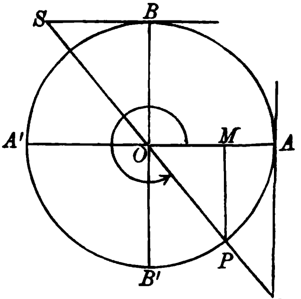

Fig. 24.

1. *The Sine.* In the first quadrant, the sine MP increases from 0 to 1; in the second, it remains positive, and decreases from 1 to 0; in the third, it is negative, and increases in absolute value from 0 to 1; in the fourth, it is negative, and decreases in absolute value from 1 to 0.

2. *The Cosine.* In the first quadrant, the cosine OM decreases from 1 to 0; in the second, it becomes negative and increases in absolute value from 0 to 1; in the third, it is negative and decreases in absolute value from 1 to 0; in the fourth, it is positive and increases from 0 to 1.

3. *The Tangent.* In the first quadrant, the tangent AT increases from 0 to ∞; in the second quadrant, as soon as the angle exceeds 90° by the smallest conceivable amount, the moving radius OP, prolonged in the direction opposite to that of OP, will cut AT at a point T situated very far *below* A; hence, the tangents of angles near 90° in the second quadrant have very large *negative* values. As the angle increases, the tangent AT continues negative but diminishes in absolute value. When $x = 180°$, then T coincides with A, and $\tan 180° = 0$. In the third quadrant, the tangent is positive and increases from 0 to ∞; in the fourth, it is negative and decreases in absolute value from ∞ to 0.

4. *The Cotangent.* In the first quadrant, the cotangent BS decreases from ∞ to 0; in the second quadrant, it is negative and increases in absolute value from 0 to ∞; in the third and fourth quadrants, it has the same sign, and undergoes the same changes as in the first and second quadrants respectively.

5. *The Secant.* In the first quadrant, the secant OT increases from 1 to ∞; in the second quadrant, it becomes negative (being measured in the direction opposite to that of OP), and decreases in absolute value from ∞ to 1, so that $\sec 180° = -1$; in the third quadrant, it continues negative, and increases in absolute value from 1 to ∞; in the fourth quadrant, it is positive, and decreases from ∞ to 1.

6. *The Cosecant.* In the first quadrant, the cosecant OS decreases from ∞ to 1; in the second quadrant, it remains positive, and increases from 1 to ∞; in the third quadrant, it becomes negative, and decreases in absolute value from ∞ to 1, so that $\csc 270° = -1$; in the fourth quadrant, it is negative, and increases in absolute value from 1 to ∞.

The limiting values of the functions are as follows:

	0°	90°	180°	270°	360°
Sine	± 0	1	± 0	− 1	± 0
Cosine	1	± 0	− 1	± 0	1
Tangent	± 0	± ∞	± 0	± ∞	± 0
Cotangent	± ∞	± 0	± ∞	± 0	± ∞
Secant	1	± ∞	− 1	± ∞	1
Cosecant	± ∞	1	± ∞	− 1	± ∞

Sines and cosines extend from $+1$ to -1; tangents and cotangents from $+\infty$ to $-\infty$; secants and cosecants from $+\infty$ to $+1$, and from -1 to $-\infty$.

In the table given above the double sign ± is placed before 0 and ∞. From the preceding investigation it appears that the functions *always change sign in passing through* 0 *and* ∞; and the sign + or − prefixed to 0 or ∞ simply shows the direction from which the value is reached.

Take, for example, $\tan 90^\circ$: The nearer an acute angle is to 90°, the greater the *positive* value of its tangent; and the nearer an obtuse angle is to 90°, the greater the *negative* value of its tangent. When the angle *is* 90°, OP (Fig. 21) is parallel to AT, and cannot meet it. But $\tan 90^\circ$ may be regarded as extending either in the positive or in the negative direction; and according to the view taken, it will be $+\infty$ or $-\infty$.

§ 26. FUNCTIONS OF ANGLES LARGER THAN 360°.

It is obvious that the functions of $360^\circ + x$ are the same both in sign and in absolute value as those of x; for the moving radius has the same position in both cases. In general, if n denote any positive whole number,

The functions of $(n \times 360^\circ + x)$ *are the same as those of* x.

For example: the functions of $2200^\circ =$ the functions of $(6 \times 360^\circ + 40^\circ) =$ the functions of 40°.

§ 27. EXTENSION OF FORMULAS [1]–[3] TO ALL ANGLES.

The Formulas established for *acute* angles in § 5 hold true for *all* angles. Thus, Formula [1],

$$\sin^2 x + \cos^2 x = 1,$$

is universally true; for, whether MP and OM (Figs. 21–24) are positive or negative, $\overline{MP}^2$ and $\overline{OM}^2$ are always positive, and in each quadrant $\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2 = 1$.

Also, Formulas

$$[2] \quad \tan x = \frac{\sin x}{\cos x},$$

and

$$[3] \begin{cases} \sin x \times \csc x = 1, \\ \cos x \times \sec x = 1, \\ \tan x \times \cot x = 1, \end{cases}$$

are universally true; for the algebraic signs of the functions, as given in the table at the end of § 24, agree with those in Formulas [2] and [3]; and with regard to the absolute values, we have in each quadrant from the similar triangles OMP, OAT, OBS, (Figs. 21–24) the proportions

$$\begin{aligned} AT : OA &= MP : OM, \\ MP : OP &= OB : OS, \\ OM : OP &= OA : OT, \\ AT : OA &= OB : BS, \end{aligned}$$

which, by substituting 1 for the radius, and the right names for the other lines, are easily reduced to the above formulas.

Formulas [1]–[3] enable us, from a given value of one function, to find the absolute values of the other five functions, and also the sign of the reciprocal function. But in order to determine the proper signs to be placed before the other four functions, we must know the quadrant to which the angle in question belongs; or what amounts to the same thing, the sign of any *one* of these four functions; for, by reference to the Table of Signs (§ 24) it will be seen that the signs of any two functions that are not reciprocals determine the quadrant to which the angle belongs.

Example. Given $\sin x = +\frac{4}{5}$, and $\tan x$ negative; find the values of the other functions.

Since $\sin x$ is positive, x must be an angle in Quadrant I. or in Quadrant II.; but, since also $\tan x$ is negative, Quadrant I. is inadmissable.

By [1], $\cos x = \pm\sqrt{1 - \frac{16}{25}} = \pm\frac{3}{5}$.

Since the angle is in Quadrant II. the minus sign must be taken, and we have

$$\cos x = -\frac{3}{5}.$$

By [2] and [3],

$$\tan x = -\frac{4}{3},\quad \cot x = -\frac{3}{4},\quad \sec x = -\frac{5}{3},\quad \csc x = \frac{5}{4}.$$

Exercise IX.

1. Construct the functions of an angle in Quadrant II. What are their signs?

2. Construct the functions of an angle in Quadrant III. What are their signs?

3. Construct the functions of an angle in Quadrant IV. What are their signs?

4. What are the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°?

5. How many angles less than 360° have the value of the sine equal to $+\frac{5}{7}$, and in what quadrants do they lie?

6. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do they lie?

7. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

8. Within what limits must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (x to be less than 360°.)

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if the cotangent is positive and the sine negative?

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive and how many negative?

11. In finding $\cos x$ by means of the equation $\cos x = \pm\sqrt{1-\sin^2 x}$, when must we choose the positive sign and when the negative sign?

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x.

15. Given $\cot x = -3$; find all the possible values of the other functions.

16. What functions of an angle of a triangle may be negative? In what case are they negative?

17. What functions of an angle of a triangle determine the angle, and what functions fail to do so?

18. Why may $\cot 360°$ be considered equal either to $+\infty$ or to $-\infty$?

19. Obtain by means of Formulas [1]–[3] the other functions of the angles given:

(i.) $\tan 90° = \infty$. (iii.) $\cot 270° = 0$.

(ii.) $\cos 180° = -1$. (iv.) $\csc 360° = -\infty$.

20. Find the values of $\sin 450°$, $\tan 540°$, $\cos 630°$, $\cot 720°$, $\sin 810°$, $\csc 900°$.

21. For what angle in each quadrant are the absolute values of the sine and cosine alike?

Compute the values of the following expressions:

22. $a \sin 0° + b \cos 90° - c \tan 180°$.

23. $a \cos 90° - b \tan 180° + c \cot 90°$.

24. $a \sin 90° - b \cos 360° + (a-b) \cos 180°$.

25. $(a^2 - b^2) \cos 360° - 4ab \sin 270°$.

§ 28. REDUCTION OF FUNCTIONS TO THE FIRST QUADRANT.

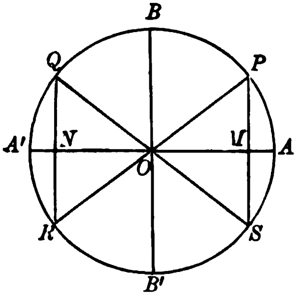

Fig. 25.

In a unit circle (Fig. 25) draw two diameters PR and QS equally inclined to the horizontal diameter AA', or so that the angles AOP, $A'OQ$, $A'OR$, and AOS shall be equal. From the points P, Q, R, S let fall perpendiculars to AA'; the four right triangles thus formed, with a common vertex at O, are equal; because they have equal hypotenuses (radii of the circle) and equal acute angles at O. Therefore, the perpendiculars PM, QN, RN, SM, are equal. Now these four lines are the sines of the angles AOP, AOQ, AOR, and AOS, respectively. Therefore, *in absolute value*,

$$\sin AOP = \sin AOQ = \sin AOR = \sin AOS.$$

And from § 27 it follows that *in absolute value* the cosines of these angles are also equal; and likewise the tangents, the cotangents, the secants, and the cosecants.*

Hence, *for every acute angle* (AOP) *there is an angle in each of the higher quadrants whose functions, in absolute value, are equal to those of this acute angle.*

Let $\angle AOP = x$, $\angle POB = y$; then $x + y = 90°$, and the functions of x are equal to the co-named functions of y (§ 4); further,

$$\begin{aligned}
\angle AOQ \text{ (in Quadrant II.)} &= 180° - x = 90° + y,\\
\angle AOR \text{ (in Quadrant III.)} &= 180° + x = 270° - y,\\
\angle AOS \text{ (in Quadrant IV.)} &= 360° - x = 270° + y.
\end{aligned}$$

Hence, if we prefix in each case the proper sign (§ 24), we have the two following series of Formulas:

---

* In future, secants and cosecants will be disregarded. They may be found by [3] if wanted, but are seldom used in computations.

Angle in Quadrant II.

$$\begin{array}{llll}
\sin(180^\circ - x) = & \sin x. & \sin(90^\circ + y) = & \cos y. \\
\cos(180^\circ - x) = & -\cos x. & \cos(90^\circ + y) = & -\sin y. \\
\tan(180^\circ - x) = & -\tan x. & \tan(90^\circ + y) = & -\cot y. \\
\cot(180^\circ - x) = & -\cot x. & \cot(90^\circ + y) = & -\tan y.
\end{array}$$

Angle in Quadrant III.

$$\begin{array}{llll}
\sin(180^\circ + x) = & -\sin x. & \sin(270^\circ - y) = & -\cos y. \\
\cos(180^\circ + x) = & -\cos x. & \cos(270^\circ - y) = & -\sin y. \\
\tan(180^\circ + x) = & \tan x. & \tan(270^\circ - y) = & \cot y. \\
\cot(180^\circ + x) = & \cot x. & \cot(270^\circ - y) = & \tan y.
\end{array}$$

Angle in Quadrant IV.

$$\begin{array}{llll}
\sin(360^\circ - x) = & -\sin x. & \sin(270^\circ + y) = & -\cos y. \\
\cos(360^\circ - x) = & \cos x. & \cos(270^\circ + y) = & \sin y. \\
\tan(360^\circ - x) = & -\tan x. & \tan(270^\circ + y) = & -\cot y. \\
\cot(360^\circ - x) = & -\cot x. & \cot(270^\circ + y) = & -\tan y.
\end{array}$$

Hence, by selecting the right formulas,

The functions of all angles can be reduced to the functions of angles not greater than 45°. Thus, to find the functions of 220° and 230°, we should consider 220° as $(180^\circ + 40^\circ)$, but 230° as $(270^\circ - 40^\circ)$.

It is evident from these formulas that,

If an acute angle be added to or subtracted from **180° or 360°,** *the functions of the resulting angle are equal in absolute value to the* **like-named** *functions of the acute angle; but if an acute angle be added to or subtracted from* **90° or 270°,** *the functions of the resulting angle are equal in absolute value to the* **co-named** *functions of the acute angle.*

It is evident from the formulas for $(180^\circ - x)$ that,

A given value of a sine determines two supplementary angles, one acute, the other obtuse; a given value of any other function (except the cosecant) determines only one angle: acute if the value is positive, obtuse if the value is negative.

§ 29. Angles whose Difference is 90°.

The general form of two such angles is x and $90° + x$, and they must lie in adjoining quadrants. The relations between their functions were found in § 28, but only for the case when x is acute. These relations, however, may be shown to hold true for all values of x.

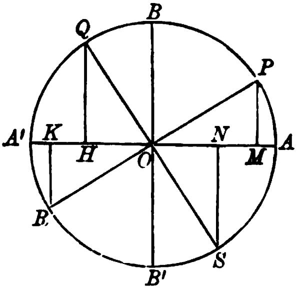

Fig. 26.

In a unit circle (Fig. 26) draw two diameters PR and QS perpendicular to each other, and let fall to AA' the perpendiculars PM, QH, RK, and SN. The right triangles OMP, OHQ, OKR, and ONS are equal, because they have equal hypotenuses and equal acute angles POM, OQH, ROK, and OSN.

Therefore, $OM = QH = OK = NS$,
and $PM = OH = KR = ON$.

Hence, taking also into account the algebraic sign,

$$\begin{array}{llll}
\sin AOQ = \cos AOP; & \sin AOS & = & \cos AOR; \\
\cos AOQ = -\sin AOP; & \cos AOS & = & -\sin AOR; \\
\sin AOR = \cos AOQ; & \sin(360° + AOP) & = & \cos AOS; \\
\cos AOR = -\sin AOQ; & \cos(360° + AOP) & = & -\sin AOS.
\end{array}$$

In all these equations, if x denote the angle on the right-hand side, the angle on the left-hand side will be $90° + x$. Therefore, if x be an angle in any one of the four quadrants,

$$\sin(90° + x) = \cos x,$$
$$\cos(90° + x) = -\sin x.$$

And, by § 27,

$$\tan(90° + x) = -\cot x,$$
$$\cot(90° + x) = -\tan x.$$

In like manner, it can be shown that all the formulas of § 28 hold true, whatever be the value of the angle x.

§ 30. Functions of a Negative Angle.

If the angle AOP (Fig. 25) is denoted by x, the equal angle AOS, generated by a *backward* rotation of the moving radius from the initial position OA, will be denoted by $-x$. It is obvious that the position OS of the moving radius for this angle is identical with its position for the angle $360° - x$. Therefore, the functions of the angle $-x$ are the same as those of the angle $360° - x$; or (§ 28),

$$\begin{aligned} \sin(-x) &= -\sin x, & \tan(-x) &= -\tan x, \\ \cos(-x) &= \cos x, & \cot(-x) &= -\cot x. \end{aligned}$$

Exercise X.

1. Express $\sin 250°$ in terms of the functions of an acute angle greater than 45°, and also in terms of the functions of an acute angle less than 45°.

Ans. 1. $\sin 250° = \sin(180° + 70°) = -\sin 70°$.
2. $\sin 250° = \sin(270° - 20°) = -\cos 20°$.

Express the following functions in terms of the functions of angles less than 45°:

2. $\sin 172°$.
3. $\cos 100°$.
4. $\tan 125°$.
5. $\cot 91°$.
6. $\sec 110°$.
7. $\csc 157°$.
8. $\sin 204°$.
9. $\cos 359°$.
10. $\tan 300°$.
11. $\cot 264°$.
12. $\sec 244°$.
13. $\csc 271°$.
14. $\sin 163° 49'$.
15. $\cos 195° 33'$.
16. $\tan 269° 15'$.
17. $\cot 139° 17'$.
18. $\sec 299° 45'$.
19. $\csc 92° 25'$.

Express all the functions of the following negative angles in terms of those of positive angles less than 45°:

20. $-75°$.
21. $-127°$.
22. $-200°$.
23. $-345°$.
24. $-52° 37'$.
25. $-196° 54'$.
26. Find the functions of 120°.

Hint. $120° = 180° - 60°$, or, $120° = 90° + 30°$; then apply § 28.

Find the functions of the following angles:

27. $135°$. 29. $210°$. 31. $240°$. 33. $-30°$.
28. $150°$. 30. $225°$. 32. $300°$. 34. $-225°$.

35. Given $\sin x = -\sqrt{\tfrac{1}{2}}$, and $\cos x$ negative; find the other functions of x, and the value of x.

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II.; find the other functions of x, and the value of x.

37. Find the functions of $3540°$.

38. What angles less than $360°$ have a sine equal to $-\tfrac{1}{2}$? a tangent equal to $-\sqrt{3}$?

39. Which of the angles mentioned in Examples 27–34 have a cosine equal to $-\sqrt{\tfrac{1}{2}}$? a cotangent equal to $-\sqrt{3}$?

40. What values of x between $0°$ and $720°$ will satisfy the equation $\sin x = +\tfrac{1}{2}$?

41. In each of the following cases find the other angle between $0°$ and $360°$ for which the corresponding function (sign included) has the same value: $\sin 12°$, $\cos 26°$, $\tan 45°$, $\cot 72°$; $\sin 191°$, $\cos 120°$, $\tan 244°$, $\cot 357°$.

42. Given $\tan 238° = 1.6$; find $\sin 122°$.

43. Given $\cos 333° = 0.89$; find $\tan 117°$.

Simplify the following expressions:

44. $a\cos(90° - x) + b\cos(90° + x)$.

45. $m\cos(90° - x)\sin(90° - x)$.

46. $(a - b)\tan(90° - x) + (a + b)\cot(90° + x)$.

47. $a^2 + b^2 - 2ab\cos(180° - x)$.

48. $\sin(90° + x)\sin(180° + x) + \cos(90° + x)\cos(180° - x)$.

49. $\cos(180° + x)\cos(270° - y) - \sin(180° + x)\sin(270° - y)$.

50. $\tan x + \tan(-y) - \tan(180° - y)$.

51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative? Represent the result by a drawing in which the sectors corresponding to the negative values are shaded.

52. Answer the question of last example for $\sin x - \cos x$.

53. Find the functions of $(x - 90°)$ in terms of the functions of x.

54. Find the functions of $(x - 180°)$ in terms of the functions of x.

§ 31. FUNCTIONS OF THE SUM OF TWO ANGLES.

In a unit circle (Fig. 27) let the angle $AOB = x$, the angle $BOC = y$; then the angle $AOC = x + y$.

In order to express $\sin(x+y)$ and $\cos(x+y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp CF$; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG =$ the angle $GDO = x$. Also,

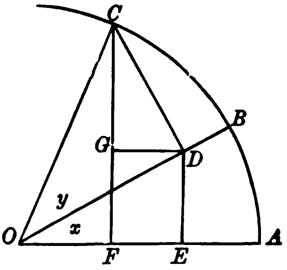

Fig. 27.

$$\sin(x+y) = CF = DE + CG.$$

$$\frac{DE}{OD} = \sin x; \quad \text{hence, } DE = \sin x \times OD = \sin x \cos y.$$

$$\frac{CG}{CD} = \cos x; \quad \text{hence, } CG = \cos x \times CD = \cos x \sin y.$$

Therefore, $\mathbf{\sin(x+y) = \sin x \cos y + \cos x \sin y.}$ [4]

Again, $\cos(x+y) = OF = OE - DG.$

$$\frac{OE}{OD} = \cos x; \quad \text{hence, } OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \quad \text{hence, } DG = \sin x \times CD = \sin x \sin y.$$

Therefore, $\mathbf{\cos(x+y) = \cos x \cos y - \sin x \sin y.}$ [5]

In this proof x and y, and also the sum $x+y$, are assumed to be acute angles. If the sum $x+y$ of the acute angles x and y is obtuse, as in Fig. 28, the proof remains, word for word, the same as above, the only difference being that the sign of OF will be negative, as DG is now greater than OE. The above formulas, therefore, hold true for all acute angles x and y.

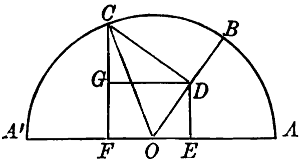

Fig. 28.

If these formulas hold true for any two acute angles x and y, they hold true when one of the angles is increased by 90°. Thus, if for x we write $x' = 90° + x$, then, by § 29,

$$\sin(x' + y) = \sin(90° + x + y) = \cos(x + y),$$
$$\cos(x' + y) = \cos(90° + x + y) = -\sin(x + y).$$

Hence, by [5], $\sin(x' + y) = \cos x \cos y - \sin x \sin y$,

by [4], $\cos(x' + y) = -\sin x \cos y - \cos x \sin y$.

Now, by § 29, $\cos x = \sin(90° + x) = \sin x'$,

$\sin x = -\cos(90° + x) = -\cos x'$.

Therefore, by putting $\sin x'$ for $\cos x$, and $-\cos x'$ for $\sin x$, in the right-hand members of the above equations,

$$\sin(x' + y) = \sin x' \cos y + \cos x' \sin y,$$
$$\cos(x' + y) = \cos x' \cos y - \sin x' \sin y.$$

Hence, it follows that Formulas [4] and [5] are universally true. For they have been proved true for any two acute angles, and also true when one of these angles is increased by 90°; hence they are true for each repeated increase of one or the other angle by 90°, and therefore true for the sum of any two angles whatever.

By § 27,

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If we divide each term of the numerator and denominator of the last fraction by $\cos x \cos y$, and again apply § 27, we obtain

$$\mathbf{\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}. \qquad [6]$$

In like manner, by dividing each term of the numerator and denominator of the value of $\cot(x + y)$ by $\sin x \sin y$, we obtain

$$\mathbf{\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}}. \qquad [7]$$

§ 32. FUNCTIONS OF THE DIFFERENCE OF TWO ANGLES.

In a unit circle (Fig. 29) let the angle $AOB = x$, the angle $COB = y$; then the angle $AOC = x - y$.

In order to express $\sin(x-y)$ and $\cos(x-y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp FC$ prolonged; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG =$ the angle $EDC = x$. And, $\sin(x-y) = CF = DE - CG$.

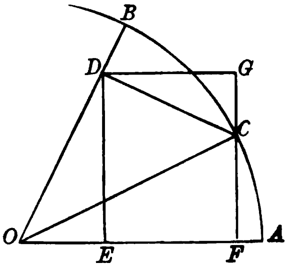

Fig. 29.

$$\frac{DE}{OD} = \sin x; \quad \text{hence, } DE = \sin x \times OD = \sin x \cos y.$$

$$\frac{CG}{CD} = \cos x; \quad \text{hence, } CG = \cos x \times CD = \cos x \sin y.$$

Therefore, $\mathbf{\sin(x - y) = \sin x \cos y - \cos x \sin y.}$ [8]

Again, $\cos(x-y) = OF = OE + DG.$

$$\frac{OE}{OD} = \cos x; \quad \text{hence, } OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \quad \text{hence, } DG = \sin x \times CD = \sin x \sin y.$$

Therefore, $\mathbf{\cos(x - y) = \cos x \cos y + \sin x \sin y.}$ [9]

In this proof, both x and y are assumed to be acute angles; but, whatever be the values of x and y, the same method of proof will always lead to Formulas [8] and [9], when due regard is paid to the algebraic signs.

The general application of these formulas may be at once shown by deducing them from the general formulas established in § 31, as follows:

It is obvious that $(x-y)+y = x$. If we apply Formulas [4] and [5] to $(x-y)+y$, then

$$\sin\{(x-y)+y\} \text{ or } \sin x = \sin(x-y)\cos y + \cos(x-y)\sin y,$$
$$\cos\{(x-y)+y\} \text{ or } \cos x = \cos(x-y)\cos y - \sin(x-y)\sin y.$$

Multiply the first equation by $\cos y$, the second by $\sin y$,

$$\sin x \cos y = \sin(x-y)\cos^2 y + \cos(x-y)\sin y \cos y,$$
$$\cos x \sin y = -\sin(x-y)\sin^2 y + \cos(x-y)\sin y\cos y;$$

whence, by subtraction,

$$\sin x \cos y - \cos x \sin y = \sin(x-y)(\sin^2 y + \cos^2 y).$$

But $\sin^2 y + \cos^2 y = 1$; therefore, by transposing,

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

Again, if we multiply the first equation by $\sin y$, the second equation by $\cos y$, and add the results, we obtain, by reducing,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Therefore, Formulas [8] and [9], like [4] and [5], from which they have been derived, are universally true.

From [8] and [9], by proceeding as in § 31, we obtain

$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}. \qquad [10]$$

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \qquad [11]$$

Formulas [4]–[11] may be combined as follows:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

§ 33. FUNCTIONS OF TWICE AN ANGLE.

If, in Formulas [4]–[7], $y - x$, they become:

$$\sin 2x = 2 \sin x \cos x. \quad [12] \qquad \cos 2x = \cos^2 x - \sin^2 x. \quad [13]$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \quad [14] \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \quad [15]$$

By these formulas the functions of twice an angle are found when the functions of the angle are given.

§ 34. FUNCTIONS OF HALF AN ANGLE.

Take the formulas

$$\cos^2 x + \sin^2 x = 1 \quad [1]$$

$$\cos^2 x - \sin^2 x = \cos 2x \quad [13]$$

Subtract, $\quad 2 \sin^2 x = 1 - \cos 2x$

Add, $\quad 2 \cos^2 x = 1 + \cos 2x$

Whence

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}, \qquad \cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}.$$

These values, if z is put for $2x$, and hence $\frac{1}{2} z$ for x, become

$$\sin \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{2}}. \quad [16] \qquad \cos \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{2}}. \quad [17]$$

Hence, by division (§ 27),

$$\tan \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{1 + \cos z}}. \quad [18] \qquad \cot \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{1 - \cos z}}. \quad [19]$$

By these formulas the functions of half an angle may be computed when the cosine of the entire angle is given.

The proper sign to be placed before the root in each case depends on the quadrant in which the angle $\frac{1}{2} z$ lies. (§ 24.)

Let the student show from Formula [18] that

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c - a}{c + a}}. \quad \text{(See page 19, Note 2.)}$$

§ 35. SUMS AND DIFFERENCES OF FUNCTIONS.

From [4], [5], [8], and [9], by addition and subtraction:

$$\begin{aligned}\sin(x+y)+\sin(x-y) &= 2\sin x\cos y,\\ \sin(x+y)-\sin(x-y) &= 2\cos x\sin y,\\ \cos(x+y)+\cos(x-y) &= 2\cos x\cos y,\\ \cos(x+y)-\cos(x-y) &= -2\sin x\sin y;\end{aligned}$$

or, by making $x+y=A$, and $x-y=B$,

and therefore, $x=\frac{1}{2}(A+B)$, and $y=\frac{1}{2}(A-B)$,

$$\mathbf{\sin A+\sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).} \qquad [20]$$

$$\mathbf{\sin A-\sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).} \qquad [21]$$

$$\mathbf{\cos A+\cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).} \qquad [22]$$

$$\mathbf{\cos A-\cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).} \qquad [23]$$

From [20] and [21], by division, we obtain

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\tan\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B);$$

or, since $\cot\frac{1}{2}(A-B)=\dfrac{1}{\tan\frac{1}{2}(A-B)}$,

$$\mathbf{\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.} \qquad [24]$$

EXERCISE XI.

1. Find the value of $\sin(x+y)$ and $\cos(x+y)$, when $\sin x = \frac{3}{5}$, $\cos x=\frac{4}{5}$, $\sin y=\frac{5}{13}$, $\cos y = \frac{12}{13}$.

2. Find $\sin(90°-y)$ and $\cos(90°-y)$ by making $x=90°$ in Formulas [8] and [9].

Find, by Formulas [4]–[11], the first four functions of:

3. $90°+y$.
4. $180°-y$.
5. $180°+y$.
6. $270°-y$.
7. $270°+y$.
8. $360°-y$.
9. $360°+y$.
10. $x-90°$.
11. $x-180°$.
12. $x-270°$.
13. $-y$.
14. $45°-y$.
15. $45°+y$.
16. $30°+y$.
17. $60°-y$.

18. Find $\sin 3x$ in terms of $\sin x$.
19. Find $\cos 3x$ in terms of $\cos x$.
20. Given $\tan \frac{1}{2}x = 1$; find $\cos x$.
21. Given $\cot \frac{1}{2}x = \sqrt{3}$; find $\sin x$.
22. Given $\sin x = 0.2$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.
23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.
24. Given $\tan 45° = 1$; find the functions of $22° 30'$.
25. Given $\sin 30° = 0.5$; find the functions of $15°$.
26. Prove that $\tan 18° = \dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°}$.

Prove the following formulas:

27. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.
29. $\tan \frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}$.
28. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.
30. $\cot \frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}$.
31. $\sin \frac{1}{2}x \pm \cos \frac{1}{2}x = \sqrt{1 \pm \sin x}$.
32. $\dfrac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y$.
33. $\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

If A, B, C are the angles of a triangle, prove that:

34. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$.
35. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$.
36. $\tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C$.
37. $\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \cot \frac{1}{2} A \times \cot \frac{1}{2} B \times \cot \frac{1}{2} C$.

Change to forms more convenient for logarithmic computation:

38. $\cot x + \tan x$.
39. $\cot x - \tan x$.
40. $\cot x + \tan y$.
41. $\cot x - \tan y$.
42. $\dfrac{1 - \cos 2x}{1 + \cos 2x}$.
43. $1 + \tan x \tan y$.
44. $1 - \tan x \tan y$.
45. $\cot x \cot y + 1$.
46. $\cot x \cot y - 1$.
47. $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

CHAPTER IV.

THE OBLIQUE TRIANGLE.

§ 36. LAW OF SINES.

Let A, B, C denote the angles of a triangle ABC (Figs. 30 and 31), and a, b, c, respectively, the lengths of the opposite sides.

Draw $CD \perp AB$, and meeting AB (Fig. 30) or AB produced (Fig. 31) at D. Let $CD = h$.

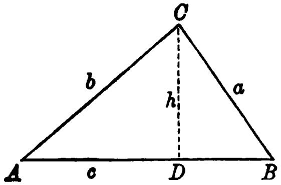

Fig. 30.

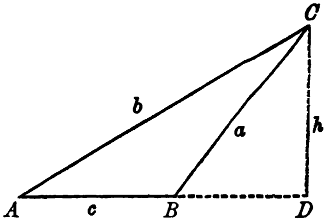

Fig. 31.

In both figures, $\frac{h}{b} = \sin A.$

In Fig. 30, $\frac{h}{a} = \sin B.$

In Fig. 31, $\frac{h}{a} = \sin (180^\circ - B) = \sin B.$

Therefore, whether h lies within or without the triangle, we obtain, by division,

$$\mathbf{\frac{a}{b} = \frac{\sin A}{\sin B}}. \qquad [25]$$

By drawing perpendiculars from the vertices A and B to the opposite sides we may obtain, in the same way,

$$\frac{b}{c}=\frac{\sin B}{\sin C}, \qquad \frac{a}{c}=\frac{\sin A}{\sin C}.$$

Hence the Law of Sines, which may be thus stated:

The sides of a triangle are proportional to the sines of the opposite angles.

If we regard these three equations as proportions, and take them by alternation, it will be evident that they may be written in the symmetrical form,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Each of these equal ratios has a simple geometrical meaning which will appear if the Law of Sines is proved as follows:

Circumscribe a circle about the triangle ABC (Fig. 32), and draw the radii OA, OB, OC; these radii divide the triangle into three isosceles triangles. Let R denote the radius. Draw $OM \perp BC$. By Geometry, the angle $BOC = 2A$; hence, the angle $BOM = A$, then $BM = R \sin BOM = R \sin A$.

$\therefore BC$ or $a = 2R \sin A$.

In like manner, $b = 2R \sin B$, and $c = 2R \sin C$. Whence we obtain

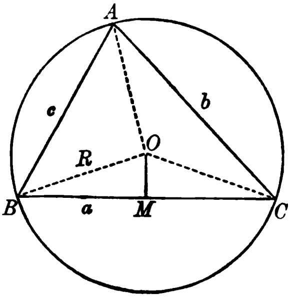

Fig. 32.

$$2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

That is: *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*

§ 37. LAW OF COSINES.

This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.

In Figs. 30 and 31, $a^2 = h^2 + BD^2$.

In Fig. 30, $BD = c - AD$;

in Fig. 31, $BD = AD - c$;

in both cases, $BD^2 = AD^2 - 2c \times AD + c^2$.

Therefore, in all cases, $a^2 = h^2 + AD^2 + c^2 - 2c \times AD$.

Now, $h^2 + AD^2 = b^2$,

and $AD = b \cos A$.

Therefore, $$\mathbf{a^2 = b^2 + c^2 - 2bc\cos A}. \qquad [26]$$

In like manner, it may be proved that

$$b^2 = a^2 + c^2 - 2ac\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The three formulas have precisely the same form, and the law may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides, diminished by twice the product of the sides and the cosine of the included angle.

§ 38. LAW OF TANGENTS.

By § 36, $a : b = \sin A : \sin B$;

whence, by the Theory of Proportion,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But by [24], page 48,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Therefore,

$$\mathbf{\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}}. \qquad [27]$$

By merely changing the letters,

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)},\qquad \frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

Hence the Law of Tangents:

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

NOTE. If in [27] $b>a$, then $B>A$. The formula is still true, but to avoid negative quantities, the formula in this case should be written

$$\frac{b-a}{b+a}=\frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}.$$

EXERCISE XII.

1. What do the formulas of § 36 become when one of the angles is a right angle?

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

3. What does Formula [26] become when $A=90°$? when $A=0°$? when $A=180°$? What does the triangle become in each of these cases?

NOTE. The case where $A=90°$ explains why the theorem of § 37 is sometimes termed the *Generalized Theorem of Pythagoras.*

4. Prove (Figs. 30 and 31) that whether the angle B is acute or obtuse $c=a\cos B+b\cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B=90°$?

5. From the three following equations (found in the last exercise) prove the theorem of § 37:

$$c=a\cos B+b\cos A,$$
$$b=a\cos C+c\cos A,$$
$$a=b\cos C+c\cos B.$$

HINT. Multiply the first equation by c, the second by b, the third by a; then from the first subtract the *sum* of the second and third.

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A-B)$? of $\frac{1}{2}(A+B)$?

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle, when

(i.) $C=90°$; (ii.) $A-B=90°$, and $B=C$.

§ 39. The Given Parts of an Oblique Triangle.

The formulas established in §§ 36–38, together with the equation $A+B+C=180°$, are sufficient for solving every case of an oblique triangle. The three parts that determine an oblique triangle may be:

I. One side and two angles;
II. Two sides and the angle opposite to one of these sides;
III. Two sides and the included angle;
IV. The three sides.

In all cases let A, B, C denote the angles, a, b, c the sides opposite these angles respectively.

§ 40. Case I.

Given one side a, and two angles A and B; find the remaining parts C, b, and c.

1. $C=180°-(A+B)$.

2. $\frac{b}{a}=\frac{\sin B}{\sin A}$; $\therefore b=\frac{a\sin B}{\sin A}=\frac{a}{\sin A}\times\sin B$.

3. $\frac{c}{a}=\frac{\sin C}{\sin A}$; $\therefore c=\frac{a\sin C}{\sin A}=\frac{a}{\sin A}\times\sin C$.

Example. $a=24.31$, $A=45°\,18'$, $B=22°\,11'$.

The work may be arranged as follows:

$a=24.31$	$\log a=1.38578$	$=1.38578$
$A=45°\,18'$	$\text{colog}\sin A=0.14825$	$=0.14825$
$B=\underline{22°\,11'}$	$\log\sin B=\underline{9.57700}$	$\log\sin C=\underline{9.96556}$
$A+B=67°\,29'$	$\log b=1.11103$	$\log c=1.49959$
$C=112°\,31'$	$b=12.913$	$c=31.593$

EXERCISE XIII.

1. Given $a = 500$, $A = 10° 12'$, $B = 46° 36'$;
find $C = 123° 12'$, $b = 2051.48$, $c = 2362.61$.

2. Given $a = 795$, $A = 79° 59'$, $B = 44° 41'$;
find $C = 55° 20'$, $b = 567.688$, $c = 663.986$:

3. Given $a = 804$, $A = 99° 55'$, $B = 45° 1'$;
find $C = 35° 4'$, $b = 577.313$, $c = 468.933$.

4. Given $a = 820$, $A = 12° 49'$, $B = 141° 59'$;
find $C = 25° 12'$, $b = 2276.63$, $c = 1573.89$.

5. Given $c = 1005$, $A = 78° 19'$, $B = 54° 27'$;
find $C = 47° 14'$, $a = 1340.6$, $b = 1113.8$.

6. Given $b = 13.57$, $B = 13° 57'$, $C = 57° 13'$;
find $A = 108° 50'$, $a = 53.276$, $c = 47.324$.

7. Given $a = 6412$, $A = 70° 55'$, $C = 52° 9'$;
find $B = 56° 56'$, $b = 5685.9$, $c = 5357.5$.

8. Given $b = 999$, $A = 37° 58'$, $C = 65° 2'$;
find $B = 77°$, $a = 630.77$, $c = 929.48$.

9. In order to determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured, and found to be 322.55 yards, 60° 34', and 56° 10', respectively. Find the distance AB.

10. In making a survey by triangulation, the angles B and C of a triangle ABC were found to be 50° 30' and 122° 9', respectively, and the length BC is known to be 9 miles. Find AB and AC.

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

12. In a parallelogram given a diagonal d and the angles x and y which this diagonal makes with the sides. Find the sides. Compute the results when $d = 11.237$, $x = 19° 1'$, and $y = 42° 54$.

13. A lighthouse was observed from a ship to bear N. 34° E.; after sailing due south 3 miles, it bore N. 23° E. Find the distance from the lighthouse to the ship in both positions.

NOTE. The phrase *to bear N. 34° E.* means that the line of sight to the lighthouse is in the north-east quarter of the horizon, and makes, with a line due north, an angle of 34°.

14. In a trapezoid given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides. Find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70°$, $y = 40°$.

Solve the following examples without using logarithms:

15. Given $b = 7.07107$, $A = 30°$, $C = 105°$; find a and c.

16. Given $c = 9.562$, $A = 45°$, $B = 60°$; find a and b.

17. The base of a triangle is 600 feet, and the angles at the base are 30° and 120°. Find the other sides and the altitude.

18. Two angles of a triangle are, the one 20°, the other 40°. Find the ratio of the opposite sides.

19. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is equal to 3. Find the other sides.

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30°. Find the radius of the circumscribed circle. (See § 36, Remark.)

§ 41. CASE II.

Given two sides a and b, and the angle A opposite to the side a; find the remaining parts B, C, c.

This case, like the preceding case, is solved by means of the Law of Sines.

Since $\dfrac{\sin B}{\sin A} = \dfrac{b}{a}$, therefore $\sin B = \dfrac{b \sin A}{a}$;

$$C = 180° - (A + B).$$

And since $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$, therefore $c = \dfrac{a \sin C}{\sin A}$.

When an angle is determined by its sine it admits of two values, which are supplements of each other (§ 28); hence, either value of B may be taken unless excluded by the conditions of the problem.

If $a > b$, then by Geometry $A > B$, and B must be acute. whatever be the value of A; for a triangle can have only one obtuse angle. Hence, there is *one, and only one, triangle* that will satisfy the given conditions.

If $a = b$, then by Geometry $A = B$; both A and B must be acute, and *the required triangle is isosceles.*

If $a < b$, then by Geometry $A < B$, and A must be acute in order that the triangle may be possible. If A is acute, it is evident from Fig. 33, where $\angle BAC = A$, $AC = b$, $CB = CB' = a$, that *the two triangles ACB and ACB' will satisfy the given conditions*, provided a is greater than the perpendicular CP; that is, provided a is greater than $b \sin A$ (§ 10). The angles ABC and $AB'C$ are supplementary (since $\angle ABC = \angle BB'C$); they are in fact the supplementary angles obtained from the formula

$$\sin B = \frac{b \sin A}{a}.$$

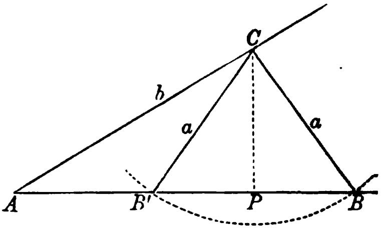

Fig. 33.

If, however, $a = b \sin A = CP$ (Fig. 33), then $\sin B = 1$, $B = 90°$, and *the triangle required is a right triangle.*

If $a < b \sin A$, that is, $< CP$, then $\sin B > 1$, and *the triangle is impossible.*

These results, for convenience, may be thus stated:

If $a > b$, or $a = b$, or if $a = b \sin A$, *One* solution.

If $a < b$, but $> b \sin A$, and $A < 90°$, *Two* solutions.

If $a < b$ and $A > 90°$, or if $a < b \sin A$ and $A < 90°$, *No* solution.

The number of solutions can often be determined by inspection. If there is any doubt, it may be removed by computing the value of $b \sin A$.

Or we may proceed to compute $\log \sin B$. If $\log \sin B = 0$, the triangle required is a right triangle. If $\log \sin B > 0$, the triangle is impossible. If $\log \sin B < 0$, there is *one* solution when $a > b$; there are *two* solutions when $a < b$.

When there are two solutions, let B', C', c' denote the unknown parts of the second triangle; then,

$$B' = 180° - B, \quad C' = 180° - (A + B') = B - A,$$

$$c' = \frac{a \sin C'}{\sin A}.$$

EXAMPLES.

1. Given $a = 16$, $b = 20$, $A = 106°$; find the remaining parts.

In this case $a < b$, and $A > 90°$; therefore the triangle is impossible.

2. Given $a = 36$, $b = 80$, $A = 30°$; find the remaining parts.

Here we have $b \sin A = 80 \times \frac{1}{2} = 40$; so that $a < b \sin A$, and the triangle is impossible.

3. Given $a = 72630$, $b = 117480$, $A = 80° 0' 50''$; find B, C, c.

$a = 72630$	$\text{colog}\, a = 5.13888$	Here $\log \sin B > 0$.
$b = 117480$	$\log b = 5.06996$	∴ *no* solution.
$A = 80° 0' 50''$	$\log \sin A = 9.99337$	
	$\log \sin B = 0.20221$	

4. Given $a = 13.2$, $b = 15.7$, $A = 57° 13' 15.3''$; find B, C, c.

$a = 13.2$	$\text{colog}\, a = 8.87943$	$c = b \cos A$
$b = 15.7$	$\log b = 1.19590$	$\log b = 1.19590$
$A = 57° 13' 15.3''$	$\log \sin A = 9.92467$	$\log \cos A = 9.73352$
Here $\log \sin B = 0$,	$\log \sin B = 0.00000$	$\log c = 0.92942$
∴ a *right* triangle.	$B = 90°$	$c = 8.5$
	$\therefore C = 32° 46' 44.7''$	

5. Given $a = 767$, $b = 242$, $A = 36° 53' 2''$; find B, C, c.

$a = 767$	$\text{colog}\, a = 7.11520$	$\log a = 2.88480$
$b = 242$	$\log b = 2.38382$	$\log\sin C = 9.86970$
$A = 36° 53' 2''$	$\log\sin A = 9.77830$	$\text{colog}\sin A = 0.22170$
Here $a > b$,	$\log\sin B = 9.27732$	$\log c = 2.97620$
and $\log\sin B < 0$.	$B = 10° 54' 58''$	$c = 946.675$
$\therefore$ *one* solution.	$\therefore C = 132° 12' 0''$	

6. Given $a = 177.01$, $b = 216.45$, $A = 35° 36' 20''$; find the other parts.

$a = 177.01$	$\text{colog}\, a = 7.75200$	$\log a = 2.24800$	2.24800
$b = 216.45$	$\log b = 2.33536$	$\text{colog}\sin A = 0.23493$	0.23493
$A = 35° 36' 20''$	$\log\sin A = 9.76507$	$\log\sin C = 9.99462$	9.23034
Here $a < b$,	$\log\sin B = 9.85243$	$\log c = 2.47755$	1.71327
and $\log\sin B < 0$.	$B = 45° 23' 28''$	$c = 300.29$ or	51.674
$\therefore$ *two* solutions.	or $134° 36' 32''$		
	$\therefore C = 99° 0' 12''$		
	or $9° 47' 8''$		

EXERCISE XIV.

1. Determine the number of solutions in each of the following cases:

(i.) $a = 80$, $b = 100$, $A = 30°$.
(ii.) $a = 50$, $b = 100$, $A = 30°$.
(iii.) $a = 40$, $b = 100$, $A = 30°$.
(iv.) $a = 13.4$, $b = 11.46$, $A = 77° 20'$.
(v.) $a = 70$, $b = 75$, $A = 60°$.
(vi.) $a = 134.16$, $b = 84.54$, $B = 52° 9' 11''$.

2. Given $a = 840$, $b = 485$, $A = 21° 31'$;
find $B = 12° 13' 34''$, $C = 146° 15' 26''$, $c = 1272.18$.

3. Given $a = 9.399$, $b = 9.197$, $A = 120° 35'$;
find $B = 57° 23' 40''$, $C = 2° 1' 20'$, $c = 0.38525$.

4. Given $a = 91.06$, $b = 77.04$, $A = 51° 9' 6''$;
find $B = 41° 13'$, $C = 87° 37' 54''$, $c = 116.82$.

5. Given $a = 55.55$, $b = 66.66$, $B = 77° 44' 40''$;
find $A = 54° 31' 13''$, $C = 47° 44' 7''$, $c = 50.481$.

6. Given $a = 309$, $b = 360$, $A = 21° 14' 25''$;
find $B = 24° 57' 54''$, $C = 133° 47' 41''$, $c = 615.67$,
$B' = 155° 2' 6''$, $C' = 3° 43' 29''$, $c' = 55.41$.

7. Given $a = 8.716$, $b = 9.787$, $A = 38° 14' 12''$;
find $B = 44° 1' 28''$, $C = 97° 44' 20''$, $c = 13.954$,
$B' = 135° 58' 32''$, $C' = 5° 47' 16''$, $c' = 1.4203$.

8. Given $a = 4.4$, $b = 5.21$, $A = 57° 37' 17''$;
find $B = 90°$, $C = 32° 22' 43''$, $c = 2.79$.

9. Given $a = 34$, $b = 22$, $B = 30° 20'$;
find $A = 51° 18' 27''$, $C = 98° 21' 33''$, $c = 43.098$,
$A' = 128° 41' 33''$, $C' = 20° 58' 27''$, $c' = 15.593$.

10. Given $b = 19$, $c = 18$, $C = 15° 49'$;
find $B = 16° 43' 13''$, $A = 147° 27' 47''$, $a = 35.519$,
$B' = 163° 16' 47''$, $A' = 0° 54' 13''$, $a' = 1.0408$,

11. Given $a = 75$, $b = 29$, $B = 16° 15' 36''$; find the difference between the areas of the two corresponding triangles.

12. Given in a parallelogram the side a, a diagonal d, and the angle A made by the two diagonals; find the other diagonal.

Special case: $a = 35$, $d = 63$, $A = 21° 36' 30''$.

§ 42. Case III.

Given two sides a and b and the included angle C; find the remaining parts A, B, and c.

Solution I. The angles A and B may both be found by means of Formula [27], § 38, which may be written

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \tfrac{1}{2}(A + B).$$

Since $\frac{1}{2}(A + B) = \frac{1}{2}(180° - C)$, the value of $\frac{1}{2}(A + B)$ is known; so that this equation enables us to find the value of $\frac{1}{2}(A - B)$. We then have

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A,$$

and

$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B.$$

After A and B are known, the side c may be found by the Law of Sines, which gives its value in two ways, as follows:

$$c = \frac{a \sin C}{\sin A}, \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

SOLUTION II. The third side c may be found directly from the equation (§ 37)

$$c = \sqrt{a^2 + b^2 - 2ab \cos C};$$

and then, by the Law of Sines, the following equations for computing the values of the angles A and B are obtained:

$$\sin A = a \times \frac{\sin C}{c}, \quad \sin B = b \times \frac{\sin C}{c}.$$

SOLUTION III. If, in the triangle ABC (Fig. 34), BD is drawn perpendicular to the side AC, then

$$\tan A = \frac{BD}{AD} = \frac{BD}{AC - DC}.$$

Now $BD = a \sin C$ (§ 10),

and $DC = a \cos C$.

$$\therefore \tan A = \frac{a \sin C}{b - a \cos C}.$$

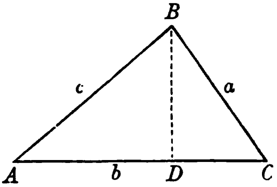

Fig. 34.

By merely changing the letters,

$$\tan B = \frac{b \sin C}{a - b \cos C}.$$

It is not necessary, however, to use both formulas. When one angle, as A, has been found, the other, B, may be found from the relation $A + B + C = 180°$.

When the angles are known, the third side is found by the Law of Sines, as in Solution I.

NOTE. When all three unknown parts are required, Solution I. is the most convenient in practice. When only the third side c is desired, Solution II. may be used to advantage, provided the values of a^2 and b^2 can be readily obtained without the aid of logarithms. But Solutions II. and III. are not adapted to logarithmic work.

EXAMPLES.

1. Given $a = 748$, $b = 375$, $C = 63° 35' 30''$; find A, B, and c.

$a + b = 1123$		
$a - b = 373$	$\log(a - b) = 2.57171$	$\log b = 2.57403$
$(A + B) = 116° 24' 30''$	$\text{colog}(a + b) = 6.94961$	$\log \sin C = 9.95214$
$\frac{1}{2}(A + B) = 58° 12' 15''$	$\log \tan \frac{1}{2}(A + B) = 0.20766$	$\text{colog} \sin B = 0.30073$
$\frac{1}{2}(A - B) = 28° 10' 52''$	$\log \tan \frac{1}{2}(A - B) = 9.72898$	$\log c = 2.82690$
$A = 86° 23' 7''$	$\frac{1}{2}(A - B) = 28° 10' 52''$	$c = 671.27$
$B = 30° 1' 23''$		

NOTE. In the above Example we use the angle B in finding the side c, rather than the angle A, because A is near 90°, and therefore its sine should be avoided.

2. Given $a = 4$, $c = 6$, $B = 60°$; find the third side b.

Here Solution II. may be used to advantage. We have

$$b = \sqrt{a^2 + c^2 - 2ac \cos B} = \sqrt{16 + 36 - 24} = \sqrt{28};$$

$$\log 28 = 1.44716, \quad \log \sqrt{28} = 0.72358, \quad \sqrt{28} = 5.2915;$$

that is, $b = 5.2915$.

EXERCISE XV.

1. Given $a = 77.99$, $b = 83.39$, $C = 72° 15'$; [§38, N.
find $A = 51° 15'$, $B = 56° 30'$, $c = 95.24$.
2. Given $b = 872.5$, $c = 632.7$, $A = 80°$;
find $B = 60° 45'$, $C = 39° 15'$, $a = 984.83$.
3. Given $a = 17$, $b = 12$, $C = 59° 17'$;
find $A = 77° 12' 53''$, $B = 43° 30' 7''$, $c = 14.987$.
4. Given $b = \sqrt{5}$, $c = \sqrt{3}$, $A = 35° 53'$;
find $B = 93° 28' 36''$, $C = 50° 38' 24''$, $a = 1.313$.
5. Given $a = 0.917$, $b = 0.312$, $C = 33° 7' 9''$;
find $A = 132° 18' 27''$, $B = 14° 34' 24''$, $c = 0.67748$.
6. Given $a = 13.715$, $c = 11.214$, $B = 15° 22' 36''$,
find $A = 118° 55' 49''$, $C = 45° 41' 35''$, $b = 4.1554$.
7. Given $b = 3000.9$, $c = 1587.2$, $A = 86° 4' 4''$;
find $B = 65° 13' 51''$, $C = 28° 42' 5''$, $a = 3297.2$.

8. Given $a = 4527$, $b = 3465$, $C = 66° 6' 27''$;
find $A = 68° 29' 15''$, $B = 45° 24' 18''$, $c = 4449$.

9. Given $a = 55.14$, $b = 33.09$, $C = 30° 24'$;
find $A = 117° 24' 33''$, $B = 32° 11' 27''$, $c = 31.431$.

10. Given $a = 47.99$, $b = 33.14$, $C = 175° 19' 10''$;
find $A = 2° 46' 8''$, $B = 1° 54' 42''$, $c = 81.066$.

11. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side.

12. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side.

13. Apply Solution I. to the case in which $a = b$ or the triangle is isosceles.

14. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

16. In order to find the distance between two objects A and B separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62° 31'$, were measured. Find the distance from A to B.

17. Two inaccessible objects A and B are each viewed from two stations C and D 562 yards apart. The angle ACB is 62° 12′, BCD 41° 8′, ADB 60° 49′, and ADC 34° 51′; required the distance AB.

18. Two trains start at the same time from the same station, and move along straight tracks that form an angle of 30°, one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form 49° 18′. Find the sides.

20. In a triangle one angle = 139° 54′, and the sides forming the angle have the ratio 5 : 9. Find the other two angles.

§ 43. Case IV.

Given the three sides a, b, c; find the angles A, B, C.

The angles may be found directly from the formulas established in § 37. Thus, from the formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

we have
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From this equation formulas adapted to logarithmic work are deduced as follows:

For the sake of brevity, let $a + b + c = 2s$; then $b + c - a = 2(s - a)$, $a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

Then the value of $1 - \cos A$ is

$$1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$
$$= \frac{(a + b - c)(a - b + c)}{2bc} = \frac{2(s - b)(s - c)}{bc};$$

and the value of $1 + \cos A$ is

$$1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2bc} = \frac{2s(s - a)}{bc}.$$

But from Formulas [16] and [17], § 34, it follows that

$1 - \cos A = 2\sin^2 \frac{1}{2} A$, and $1 + \cos A = 2\cos^2 \frac{1}{2} A$.

$\therefore\ 2\sin^2 \frac{1}{2} A = \frac{2(s - b)(s - c)}{bc}$, and $2\cos^2 \frac{1}{2} A = \frac{2s(s - a)}{bc}$,

whence
$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}, \qquad [28]$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}, \qquad [29]$$

and therefore
$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \qquad [30]$$

By merely changing the letters,

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

There is then a choice of three different formulas for finding the value of each angle. If half the angle is very near 0°, the formula for the cosine will not give a very accurate result, because the cosines of angles near 0° differ little in value; and the same holds true of the formula for the sine when half the angle is very near 90°. Hence, in the first case the formula for the sine, in the second that for the cosine, should be used.

But, in general, the formulas for the tangent are to be preferred.

It is not necessary to compute by the formulas more than two angles; for the third may then be found from the equation

$$A + B + C = 180°.$$

There is this advantage, however, in computing all three angles by the formulas, that we may then use the sum of the angles as a test of the accuracy of the results.

In case it is desired to compute all the angles, the formulas for the tangent may be put in a more convenient form.

The value of tan $\frac{1}{2}A$ may be written

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} \text{ or } \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence, if we put

$$\sqrt{\frac{(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{s}}} = \mathbf{r}, \qquad [31]$$

we have

$$\tan \tfrac{1}{2} \mathbf{A} = \frac{\mathbf{r}}{\mathbf{s}-\mathbf{a}}. \qquad [32]$$

In like manner,

$$\tan \tfrac{1}{2} B = \frac{r}{s-b}, \qquad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

EXAMPLES.

1. Given $a = 3.41$, $b = 2.60$, $c = 1.58$; find the angles.

Using Formula [30], and the corresponding formula for $\tan \frac{1}{2} B$, we may arrange the work as follows:

$a = 3.41$	$\operatorname{colog} s = 9.42079$	$\operatorname{colog} s = 9.42079 - 10$
$b = 2.60$	$\operatorname{colog}(s-a) = 0.41454$	$\log(s-a) = 9.58546 - 10$
$c = 1.58$	$\log(s-b) = 0.07737$	$\operatorname{colog}(s-b) = 9.92263 - 10$
$2s = 7.59$	$\log(s-c) = 0.34537$	$\log(s-c) = 0.34537$
$s = 3.795$	$2)0.25807$	$2)19.27425 - 20$
$s-a = 0.385$	$\log\tan \frac{1}{2}A = 0.12903$	$\log\tan \frac{1}{2}B = 9.63713 - 10$
$s-b = 1.195$	$\frac{1}{2}A = 53° 23' 20''$	$\frac{1}{2}B = 23° 26' 37''$
$s-c = 2.215$	$A = 106° 46' 40''$	$B = 46° 53' 14''$

$\therefore A + B = 153° 39' 54''$, and $C = 26° 20' 6''$.

2. Solve Example 1 by finding all three angles by the use of Formulas [31] and [32].

Here the work may be compactly arranged as follows, if we find $\log\tan \frac{1}{2}A$, etc., by *subtracting* $\log(s-a)$, etc., from $\log r$ instead of adding the cologarithm:

$a = 3.41$	$\log(s-a) = 9.58546$	$\log\tan \frac{1}{2}A = 10.12903$
$b = 2.60$	$\log(s-b) = 0.07737$	$\log\tan \frac{1}{2}B = 9.63713$
$c = 1.58$	$\log(s-c) = 0.34537$	$\log\tan \frac{1}{2}C = 9.36912$
$2s = 7.59$	$\operatorname{colog} s = 9.42079$	$\frac{1}{2}A = 53° 23' 20''$
$s = 3.795$	$\log r^2 = 9.42899$	$\frac{1}{2}B = 23° 26' 37''$
$s-a = 0.385$	$\log r = 9.71450$	$\frac{1}{2}C = 13° 10' 3''$
$s-b = 1.195$		$A = 106° 46' 40''$
$s-c = 2.215$		$B = 46° 53' 14''$
$2s = 7.590$ (proof).		$C = 26° 20' 6''$

Proof, $A + B + C = 180° 0' 0''$

NOTE. Even if no mistakes are made in the work the sum of the three angles found as above may differ very slightly from 180° in consequence of the fact that logarithmic computation is at best only a method of close approximation. When a difference of this kind exists it should be divided among the angles according to the probable amount of error for each angle.

EXERCISE XVI.

Solve the following triangles, taking the three sides as the given parts:

	a	b	c	A	B	C
1	51	65	20	38° 52′ 48″	126° 52′ 12″	14° 15′
2	78	101	29	32° 10′ 54″	136° 23′ 50″	11° 25′ 16″
3	111	145	40	27° 20′ 32″	143° 7′ 48″	9° 31′ 40″
4	21	26	31	42° 6′ 13″	56° 6′ 36″	81° 47′ 11″
5	19	34	49	16° 25′ 36″	30° 24′	133° 10′ 24″
6	43	50	57	46° 49′ 35″	57° 59′ 44″	75° 10′ 41″
7	37	58	79	26° 0′ 29″	43° 25′ 20″	110° 34′ 11″
8	73	82	91	49° 34′ 58″	58° 46′ 58″	71° 38′ 4″
9	14.493	55.4363	66.9129	8° 20′	33° 40′	138°
10	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	51° 53′ 12″	59° 31′ 48″	68° 35′

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.
12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.
13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.
14. Given $a = 6$, $b = 5$, $c = 12$; find the angles.
15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.
16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.

17. The distances between three cities A, B, and C are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A. In what direction is C from A? What two answers are admissible?

18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?

19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II. not exist?

20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.

21. Of three towns A, B, and C, A is 200 miles from B and 184 miles from C, B is 150 miles due north from C; how far is A north of C?

§ 44. AREA OF A TRIANGLE.

If F denote the area of the triangle ABC (Fig. 30 or 31, page 50), then, by Geometry,

$$F = \tfrac{1}{2} c \times CD.$$

By § 10, $CD = a \sin B$.

Therefore, $\mathbf{F = \tfrac{1}{2} ac \sin B.}$ [33]

And, in like manner,

$$F = \tfrac{1}{2} ab \sin C \quad \text{and} \quad F = \tfrac{1}{2} bc \sin A.$$

That is: *The area of a triangle is equal to half the product of two sides and the sine of the included angle.*

By Formula [33] the area of a triangle may be found directly when two sides and the included angle are given; in the other cases the formula may be used when these parts have been computed.

When the three sides of a triangle are given, as in Case IV., a formula for its area may be found as follows:

By § 33, $\sin B = 2 \sin \tfrac{1}{2} B \times \cos \tfrac{1}{2} B$.

By substituting for $\sin \tfrac{1}{2} B$ and $\cos \tfrac{1}{2} B$ their values in terms of the sides given in § 43,

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

By substituting this value of $\sin B$ in [33],

$$\mathbf{F = \sqrt{s(s-a)(s-b)(s-c)}.} \qquad [34]$$

If R denote (as in § 36) the radius of the circumscribed circle, we have, from § 36,

$$\sin B = \frac{b}{2R}.$$

By substituting this value of $\sin B$ in [33],

$$\mathbf{F = \frac{abc}{4R}.} \qquad [35]$$

If r denote the radius of the inscribed circle, and we divide the triangle into three triangles by lines from the centre of this circle to the vertices, the altitude of each of the three triangles is equal to r. Therefore,

$$\mathbf{F} = \tfrac{1}{2}\mathbf{r}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \mathbf{rs}. \qquad [36]$$

By substituting in this formula the value of F given in [34],

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

whence r, in [31] § 43, is equal to the radius of the inscribed circle.

Exercise XVII.

Find the area:

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116° 30' 20''$.
2. Given $b = 21.66$, $c = 36.94$, $A = 66° 4' 19''$.
3. Given $a = 510$, $c = 173$, $B = 162° 30' 28''$.
4. Given $a = 408$, $b = 41$, $c = 401$.
5. Given $a = 40$, $b = 13$, $c = 37$.
6. Given $a = 624$, $b = 205$, $c = 445$.
7. Given $b = 149$, $A = 70° 42' 30''$, $B = 39° 18' 28''$.
8. Given $a = 215.9$, $c = 307.7$, $A = 25° 9' 31''$.
9. Given $b = 8$, $c = 5$, $A = 60°$.
10. Given $a = 7$, $c = 3$, $A = 60°$.
11. Given $a = 60$, $B = 40° 35' 12''$, area $= 12$; find the radius of the inscribed circle.
12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.
13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.
14. Two sides and included angle of a triangle are 2416, 1712, and 30°; and two sides and included angle of another triangle are 1948, 2848, and 150°; find the sum of their areas.
15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

EXERCISE XVIII.

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. 33° 45′ W.; and after the ship had sailed 18 miles S. 67° 30′ W. it bore N. 11° 15′ E. Find its distance from each position of the ship.

2. Two objects, A and B, were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed north-west 5 miles, when it was found that A bore due east and B bore north-east. Find the distance from A to B.

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80°; the height of the castle is 140 feet. Find the height of the monument.

4. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

5. If the sun's altitude is 30°, find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5. (Two solutions.)

7. A and B, two inaccessible objects in the same horizontal plane, are observed from a balloon at C and from a point D directly under the balloon, and in the same horizontal plane with A and B. If $CD = 2000$ yards, $\angle\ ACD = 10°\ 15'\ 10''$, $\angle\ BCD = 6°\ 7'\ 20''$, $\angle\ ADB = 49°\ 34'\ 50''$, find AB.

8. A and B are two objects whose distance, on account of intervening obstacles, cannot be directly measured. At the summit C of a hill, whose height above the common horizontal plane of the objects is known to be 517.3 yards, $\angle\ ACB$ is found to be $15°\ 13'\ 15''$. The angles of elevation of C viewed from A and B are $21°\ 9'\ 18''$ and $23°\ 15'\ 34''$ respectively. Find the distance from A to B.

MISCELLANEOUS PROBLEMS.

[Selected by permission from "Problems in Plane Trigonometry," prepared by Prof. C. J. White, of Harvard College, and published by Charles W. Sever, Cambridge.]

1. The *angular distance* of any object from a horizontal plane, as observed at any point of that plane, is the angle which a line drawn from the object to the point of observation makes with the plane. If the object observed be situated above the horizontal plane (that is, if it is farther from the earth's centre than the plane is), its angular distance from the plane is called its *angle of elevation.* If the object be below the plane, its angular distance from the plane is called its *angle of depression.* These angles are evidently vertical angles.

If two objects are in the same horizontal plane with the point of observation, the angular distance of one object from the other is called its *bearing* from that object.

If two objects are not in the same horizontal plane with either each other or the point of observation, we may suppose vertical lines to be passed through the two objects, and to meet the horizontal plane of the point of observation in two points. The angular distance of these two points is the bearing of either of the objects from the other. It may also be called the *horizontal distance* of one object from the other.

NOTE. "Problems in Plane Trigonometry" can be obtained in pamphlet form of Charles W. Sever, Cambridge, Mass.

RIGHT TRIANGLES.

2. The angle of elevation of a tower is 48° 19′ 14″, and the distance of its base from the point of observation is 95 ft. Find the height of the tower, and the distance of its top from the point of observation.

3. From a mountain 1000 ft. high, the angle of depression of a ship is 77° 35′ 11″. Find the distance of the ship from the summit of the mountain.

4. A flag-staff 90 ft. high, on a horizontal plane, casts a shadow of 117 ft. Find the altitude of the sun.

5. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is 57′ 3″. If the earth's radius is 3956.2 miles, what is the moon's distance from the earth's centre?

6. The angle at the earth's centre subtended by the sun's radius is 16′ 2″, and the sun's distance is 92,400,000 miles. Find the sun's diameter in miles.

7. The latitude of Cambridge, Mass., is 42° 22′ 49″. What is the length of the radius of that parallel of latitude?

8. At what latitude is the circumference of the parallel half of that of the equator?

9. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

10. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

11. A tower 93.97 ft. high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 25° 12′ 54″. Find the breadth of the river.

12. From a tower 58 ft. high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are 30° 13′ 18″ and 45° 46′ 14″. Find the distance between these two objects.

13. Standing directly in front of one corner of a flat-roofed house, which is 150 ft. in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{5}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

14. A regular pyramid, with a square base, has an edge 150 ft. in length, and the length of a side of its base is 200 ft. Find the inclination of the face of the pyramid to the base.

15. From one edge of a ditch 36 ft. wide, the angle of elevation of a wall on the opposite edge is 62° 39′ 10″. Find the length of a ladder which will reach from the point of observation to the top of the wall.

16. The top of a flag-staff has been broken off, and touches the ground at a distance of 15 ft. from the foot of the staff. The length of the broken part being 39 ft., find the whole length of the staff.

17. From a balloon, which is directly above one town, is observed the angle of depression of another town, 10° 14′ 9″. The towns being 8 miles apart, find the height of the balloon.

18. From the top of a mountain 3 miles high the angle of depression of the most distant object which is visible on the earth's surface is found to be 2° 13′ 50″. Find the diameter of the earth.

19. A ladder 40 ft. long reaches a window 33 ft. high, on one side of a street. Being turned over upon its foot, it reaches another window 21 ft. high, on the opposite side of the street. Find the width of the street.

20. The height of a house subtends a right angle at a window on the other side of the street; and the elevation of the top of the house, from the same point, is 60°. The street is 30 ft. wide. How high is the house?

21. A lighthouse 54 ft. high is situated on a rock. The elevation of the top of the lighthouse, as observed from a ship, is 4° 52′, and the elevation of the top of the rock is 4° 2′. Find the height of the rock, and its distance from the ship.

22. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be 35° 30′; the balloon drifts $2\frac{1}{2}$ miles east at the same height, when the angle of depression of the same object is 23° 14′. Find the height of the balloon.

23. A man standing south of a tower, on the same horizontal plane, observes its elevation to be 54° 16′: he goes east 100 yds., and then finds its elevation is 50° 8′. Find the height of the tower.

24. The elevation of a tower at a place A south of it is 30°; and at a place B, west of A, and at a distance of a from it, the the elevation is 18°. Show that the height of the tower is $\dfrac{a}{\sqrt{(2+2\sqrt{5})}}$; the tangent of 18° being $\dfrac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}$.

25. A pole is fixed on the top of a mound, and the angles of elevation of the top and the bottom of the pole are 60° and 30°. Prove that the length of the pole is twice the height of the mound.

26. At a distance (a) from the foot of a tower, the angle of elevation (A) of the top of the tower is the complement of the angle of elevation of a flag-staff on top of it. Show that the length of the staff is $2a \cot 2A$.

27. *A line of true level* is a line every point of which is equally distant from the centre of the earth. A line drawn

tangent to a line of true level at any point is a line of *apparent* level. If at any point both these lines be drawn, and extended one mile, find the distance they are then apart.

28. In Problem 2, determine the effect upon the computed height of the tower, of an error in either the angle of elevation or the measured distance.

OBLIQUE TRIANGLES.

29. To determine the height of an inaccessible object situated on a horizontal plane, by observing its angles of elevation at two points in the same line with its base, and measuring the distance of these two points.

30. The angle of elevation of an inaccessible tower, situated on a horizontal plane, is 63° 26′; at a point 500 ft. farther from the base of the tower the elevation of its top is 32° 14′. Find the height of the tower.

31. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60° 13′, and from a point 40 ft. more distant the elevation is 50° 19′. Find the breadth of the river.

32. A ship sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S.S.W. Find the ship's rate.

33. To determine the height of an accessible object situated on an inclined plane.

34. At a distance of 40 ft. from the foot of a tower on an inclined plane, the tower subtends an angle of 41° 19′; at a point 60 ft. farther away, the angle subtended by the tower is 23° 45′. Find the height of the tower.

35. A tower makes an angle of 113° 12′ with the inclined plane on which it stands; and at a distance of 89 ft. from its base, measured down the plane, the angle subtended by the tower is 23° 27′. Find the height of the tower.

36. From the top of a house 42 ft. high, the angle of elevation of the top of a pole is 14° 13′; at the bottom of the house it is 23° 19′. Find the height of the pole.

37. The sides of a triangle are 17, 21, 28; prove that the length of a line bisecting the greatest side and drawn from the opposite angle is 13.

38. A privateer, 10 miles S.W. of a harbor, sees a ship sail from it in a direction S. 80° E., at a rate of 9 miles an hour. In what direction, and at what rate, must the privateer sail in order to come up with the ship in 1¼ hours?

39. A person goes 70 yds. up a slope of 1 in 3¼ from the edge of a river, and observes the angle of depression of an object on the opposite shore to be 2¼°. Find the breadth of the river.

40. The length of a lake subtends, at a certain point, an angle of 46° 24′, and the distances from this point to the two extremities of the lake are 346 and 290 ft. Find the length of the lake.

41. Two ships are a mile apart. The angular distance of the first ship from a fort on shore, as observed from the second ship, is 35° 14′ 10″; the angular distance of the second ship from the fort, observed from the first ship, is 42° 11′ 53″. Find the distance in feet from each ship to the fort.

42. Along the bank of a river is drawn a base line of 500 feet. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 53°; that of the second extremity from the same object, observed at the first, is 79° 12′. Find the perpendicular breadth of the river.

43. A vertical tower stands on a declivity inclined 15° to the horizon. A man ascends the declivity 80 ft. from the base of the tower, and finds the angle then subtended by the tower to be 30°. Find the height of the tower.

44. The angle subtended by a tower on an inclined plane is, at a certain point, 42° 17′; 325 ft. farther down, it is 21° 47′. The inclination of the plane is 8° 53′. Find the height of the tower.

45. A cape bears north by east, as seen from a ship. The ship sails northwest 30 miles, and then the cape bears east. How far is it from the second point of observation?

46. Two observers, stationed on *opposite* sides of a cloud, observe its angles of elevation to be 44° 56′ and 36° 4′. Their distance from each other is 700 ft. What is the linear height of the cloud?

47. From a point B at the foot of a mountain, the elevation of the top A is 60°. After ascending the mountain one mile, at an inclination of 30° to the horizon, and reaching a point C, the angle ACB is found to be 135°. Find the height of the mountain in feet.

48. From a ship two rocks are seen in the same right line with the ship, bearing N. 15° E. After the ship has sailed northwest 5 miles, the first rock bears east, and the second northeast. Find the distance between the rocks.

49. From a window on a level with the bottom of a steeple the elevation of the steeple is 40°, and from a second window 18 ft. higher the elevation is 37° 30′. Find the height of the steeple.

50. To determine the distance between two inaccessible objects by observing angles at the extremities of a line of known length.

51. Wishing to determine the distance between a church A and a tower B, on the opposite side of a river, I measure a line CD along the river (C being nearly opposite A), and observe the angles ACB, 58° 20′; ACD, 95° 20′; ADB, 53° 30′; BDC, 98° 45′. CD is 600 ft. What is the distance required?

52. Wishing to find the height of a summit A, I measure a horizontal base line CD, 440 yds. At C, the elevation of A is 37° 18′, and the horizontal angle between D and the summit is 76° 18′; at D, the horizontal angle between C and the summit is 67° 14′. Find the height.

53. A balloon is observed from two stations 3000 ft. apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon.

54. Two forces, one of 410 pounds, and the other of 320 pounds, make an angle of 51° 37′. Find the intensity and the direction of their resultant.

55. An unknown force, combined with one of 128 pounds, produces a resultant of 200 pounds, and this resultant makes an angle of 18° 24′ with the known force. Find the intensity and direction of the unknown force.

56. At two stations, the height of a kite subtends the same angle (A). The angle which the line joining one station and the kite subtends at the other station is B; and the distance between the two stations is a. Show that the height of the kite is $\frac{1}{2} a \sin A \sec B$.

57. Two towers on a horizontal plane are 120 ft. apart. A person standing successively at their bases observes that the angular elevation of one is double that of the other; but, when he is half-way between them, the elevations are complementary. Prove that the heights of the towers are 90 and 40 ft.

58. To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and join AB, Ab, and Ba. Measure AB, 500; aA, 100; aB, 560; bB, 100; and Ab, 550.

59. Two inaccessible points *A* and *B*, are visible from *D*, but no other point can be found whence both are visible. Take some point *C*, whence *A* and *D* can be seen, and measure *CD* 200 ft.; *ADC*, 89°; *ACD*, 50° 30′. Then take some point *E*, whence *D* and *B* are visible, and measure *DE*, 200; *BDE*, 54° 30′; *BED*, 88° 30′. At *D* measure *ADB*, 72° 30′. Compute the distance *AB*.

60. To compute the horizontal distance between two inaccessible points *A* and *B*, when no point can be found whence both can be seen. Take two points *C* and *D*, distant 200 yds. so that *A* can be seen from *C*, and *B* from *D*. From *C* measure *CF*, 200 yds. to *F*, whence *A* can be seen; and from *D* measure *DE*, 200 yds. to *E*, whence *B* can be seen. Measure *AFC*, 83°; *ACD*, 53° 30′; *ACF*, 54° 31′; *BDE*, 54° 30′; *BDC*, 156° 25′; *DEB*, 88° 30′.

61. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 ft. in length, and the elevation of the column, at the observer's station, is 45°. Find the height of the column.

62. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30°; and the horizontal angle between the two objects is 30°. Show that the height of the hill equals the distance between the objects.

63. Wishing to know the breadth of a river from *A* to *B*, I take *AC*, 100 yds. in the prolongation of *BA*, and then take *CD*, 200 yds. at right angles to *AC*. The angle *BDA* is 37° 18′ 30″. Find *AB*.

64. The sum of the sides of a triangle is 100. The angle at *A* is double that of *B*, and the angle at *B* is double that at *C*. Determine the sides.

65. If $\sin^2 A + 5\cos^2 A = 3$, find A.

66. If $\sin^2 A = m\cos A - n$, find $\cos A$.

67. Given $\sin A = m\sin B$, and $\tan A = n\tan B$, find $\sin A$ and $\cos B$.

68. If $\tan^2 A + 4\sin^2 A = 6$, find A.

69. If $\sin A = \sin 2A$, find A.

70. If $\tan 2A = 3\tan A$, find A.

71. Prove that $\tan 50^\circ + \cot 50^\circ = 2\sec 10^\circ$.

72. Given a regular polygon of n sides, and calling one of them a, find expressions for the radii of the inscribed and the circumscribed circle in terms of n and a.

If P, H, D be the sides of a regular inscribed pentagon, hexagon, and decagon, prove $P^2 = H^2 + D^2$.

AREAS.

73. Obtain the formula for the area of a triangle, given two sides b, c, and the included angle A.

74. Obtain the formula for the area of a triangle, given two angles A and B, and included side c.

75. Obtain the formula for the area of a triangle, given the three sides.

76. If a is the side of an equilateral triangle, its area is $\frac{a^2\sqrt{3}}{4}$.

77. Two consecutive sides of a rectangle are 52.25 ch. and 38.24 ch. Find its area.

78. Two sides of a parallelogram are 59.8 ch. and 37.05 ch., and the included angle is 72° 10′. Find the area.

79. Two sides of a parallelogram are 15.36 ch. and 11.46 ch., and the included angle is 47° 30′. Find its area.

80. Two sides of a triangle are 12.38 ch. and 6.78 ch., and the inclined angle is 46° 24′. Find the area.

81. Two sides of a triangle are 18.37 ch. and 13.44 ch., and they form a right angle. Find the area.

82. Two angles of a triangle are 76° 54′ and 57° 33′ 12″, and the included side is 9 ch. Find the area.

83. Two sides of a triangle are 19.74 ch. and 17.34 ch. The first bears N. 82° 30′ W.; the second, S. 24° 15′ E. Find the area.

84. The three sides of a triangle are 49 ch., 50.25 ch., and 25.69 ch. Find the area.

85. The three sides of a triangle are 10.64 ch., 12.28 ch., and 9 ch. Find the area.

86. The sides of a triangular field, of which the area is 14 acres, are in the ratio of 3, 5, 7. Find the sides.

87. In the quadrilateral $ABCD$ we have AB, 17.22 ch.; AD, 7.45 ch.; CD, 14.10 ch.; BC, 5.25 ch.; and the diagonal AC, 15.04 ch. Required the area.

88. The diagonals of a quadrilateral are a and b, and they intersect at an angle D. The area of the quadrilateral is $\frac{1}{2}ab\sin D$.

89. The diagonals of a quadrilateral are 34 and 56, intersecting at an angle of 67°. Find the area.

90. The diagonals of a quadrilateral are 75 and 49, intersecting at an angle of 42°. Find the area.

91. The area of a regular polygon of n sides, of which one is a, is $\frac{na^2}{4}\cot\frac{180^\circ}{n}$.

92. One side of a regular pentagon is 25. Find the area.

93. One side of a regular hexagon is 32. Find the area.

94. One side of a regular decagon is 46. Find the area.

95. Find the area of a circle whose circumference is 74 ft.

96. Find the area of a circle whose radius is 125 ft.

97. In a circle with a diameter of 125 ft. find the area of a sector with an arc of 22°.

98. In a circle with a radius of 44 ft. find the area of a sector with an arc of 25°.

99. In a circle with a diameter of 50 ft. find the area of a segment with an arc of 280°.

100. Find the area of a segment (less than a semicircle), of which the chord is 20, and the distance of the chord from the middle point of the smaller arc is 2.

101. If r is the radius of a circle, the area of a regular circumscribed polygon of n sides is $nr^2 \tan \frac{180^\circ}{n}$.

The area of a regular inscribed polygon is $\frac{n}{2} r^2 \sin \frac{360^\circ}{n}$.

102. If a is a side of a regular polygon of n sides, the area of the inscribed circle is $\frac{\pi a^2}{4} \cot^2 \frac{180^\circ}{n}$.

The area of the circumscribed circle is $\frac{\pi a^2}{4} \csc^2 \frac{180^\circ}{n}$.

103. The area of a regular polygon inscribed in a circle is to that of the circumscribed polygon of the same number of sides as 3 to 4. Find the number of sides.

104. The area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

105. The area of a circumscribed regular polygon is an harmonic mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.

106. The perimeter of a circumscribed regular triangle is double that of the inscribed regular triangle.

107. The square described about a circle is four-thirds the inscribed dodecagon.

108. Two sides of a triangle are 3 and 12, and the included angle is 30°. Find the hypotenuse of an isosceles right triangle of equal area.

PLANE SAILING.

109. **Plane Sailing** is that branch of Navigation in which the surface of the earth is considered a plane. The problems which arise are therefore solved by the methods of Plane Trigonometry.

The following definitions will explain the technical terms which are employed :

The *difference of latitude* of two places is the arc of a meridian comprehended between the parallels of latitude passing through those places.

The *departure* between two meridians is the arc of a parallel of latitude comprehended between those meridians. It evidently diminishes as the distance from the equator at which it is measured increases.

When a ship sails in such a manner as to cross successive meridians at the same angle, it is said to sail on a *rhumb-line.* The constant angle which this line makes with the meridians is called the *course,* and the *distance* between two places is measured on a rhumb-line.

If we neglect the curvature of the earth, and consider the distance, departure, and difference of latitude of two places to be straight lines, lying in one plane, they will form a right triangle, called *the triangle of plane sailing.* If ABD be a plane triangle, right-angled at D, and AD represent the difference of latitude of A and B, DAB will be the course from

A to B, AB the distance, and DB the departure, measured from B, between the meridian of A and that of B.

110. Taking the earth's equatorial diameter to be 7925.6 miles, find the length in feet of the arc of one minute of a great circle.*

111. A ship sails from latitude 43° 45′ S., on a course N. by E., 2345 miles. Find the latitude reached, and the departure made.

112. A ship sails from latitude 1° 45′ N., on a course S.E. by E., and reaches latitude 2° 31′ S. Find the distance, and the departure.

113. A ship sails from altitude 13° 17′ S., on a course N.E. by E. ¾ E., until the departure is 207 miles. Find the distance, and the latitude reached.

114. A ship sails on a course between S. and E., 244 miles, leaving latitude 2° 52′ S., and reaching latitude 5° 8′ S. Find the course, and the departure.

115. A ship sails from latitude 32° 18′ N., on a course between N. and W., making a distance of 344 miles, and a departure of 103 miles. Find the course, and the latitude reached.

116. A ship sails on a course between S. and E., making a difference of latitude 136 miles, and a departure 203 miles. Find the distance, and the course.

117. A ship sails due north 15 *statute* miles an hour, for one day. What is the distance, in a straight line, from the point left to the point reached? (Take earth's radius, 3962.8 statute miles.)

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* The length of the arc of one minute of a great circle of the earth is called a *geographical mile*, or a *knot*. In the following problems, this is the distance meant by the term "mile," unless otherwise stated.

PARALLEL AND MIDDLE LATITUDE SAILING.

118. The difference of longitude of two places is the angle at the pole made by the meridians of these two places; or, it is the arc of the equator comprehended between these two meridians.

119. In **Parallel Sailing,** a vessel is supposed to sail on a parallel of latitude; that is, either due east or due west. The distance sailed is, in this case, evidently the departure made; and the difference of longitude made depends on the solution of the following problem:

120. Given the departure between any two meridians at any latitude, find the angle which those meridians make, or the difference of longitude of any point on one meridian from any point on the other. (The earth is considered to be a perfect sphere, and the solution depends on simple geometric and trigonometric principles. *Cf.* Problem 7.) The solution gives the following formula:

Diff. long. = depart. × sec. lat.

121. A ship in latitude 42° 16′ N., longitude 72° 16′ W., sails due east a distance of 149 miles. What is the position of the point reached?

122. A ship in latitude 44° 49′ S., longitude 119° 42′ E., sails due west until it reaches longitude 117° 16′ E. Find the distance made.

123. In **Middle Latitude Sailing,** the departure between two places, not on the same parallel of latitude, is considered to be, approximately, the departure between the meridians of those places, measured on that parallel of latitude which lies midway between the parallels of the two places. Except in very high latitudes or excessive runs, such an assumption produces no great error. By the formula of Art. 120, then, we shall have —

Diff. long. = depart. × sec. mid. lat.

124. A ship leaves latitude 31° 14′ N., longitude 42° 19′ W., and sails E.N.E. 325 miles. Find the position reached.

125. Find the bearing and distance of Cape Cod from Havana. (Cape Cod, 42° 2′ N., 70° 3′ W.; Havana, 23° 9′ N., 82° 22′ W.)

126. Leaving latitude 49° 57′ N., longitude 15° 16′ W., a ship sails between S. and W. till the departure is 194 miles, and the latitude is 47° 18′ N. Find the course, distance, and longitude reached.

127. Leaving latitude 42° 30′ N., longitude 58° 51′ W., a ship sails S.E. by S. 300 miles. Find the position reached.

128. Leaving latitude 49° 57′ N., longitude 30° W., a ship sails S. 39° W., and reaches latitude 47° 44′ N. Find the distance, and longitude reached.

129. Leaving latitude 37° N., longitude 32° 16′ W., a ship sails between N. and W. 300 miles, and reaches latitude 41° N. Find the course, and longitude reached.

130. Leaving latitude 50° 10′ S., longitude 30° E., a ship sails E.S.E., making 160 miles' departure. Find the distance, and position reached.

131. Leaving latitude 49° 30′ N., longitude 25° W., a ship sails between S. and E. 215 miles, making a departure of 167 miles. Find the course, and position reached.

132. Leaving latitude 43° S., longitude 21° W., a ship sails 273 miles, and reaches latitude 40° 17′ S. What are the *two* courses and longitudes, either one of which will satisfy the data?

133. Leaving latitude 17° N., longitude 119° E., a ship sails 219 miles, making a departure of 162 miles. What four sets of answers do we get?

134. A ship in latitude 30° sails due east 360 statute miles. What is the shortest distance from the point left to the point reached ?

Solve the same problem for latitude 45°, 60°, etc.

TRAVERSE SAILING.

135. **Traverse Sailing** is the application of the principles of Plane and Middle Latitude Sailing to cases when the ship sails from one point to another on two or more different courses. Each course is worked up by itself, and these independent results are combined, as may be seen in the solution of the following example :

136. Leaving latitude 37° 16′ S., longitude 18° 42′ W., a ship sails N.E. 104 miles, then N.N.W. 60 miles, then W. by S. 216 miles. Find the position reached, and its bearing and distance from the point left.

We have, for the first course, difference of latitude 73.5 N., departure 73.5 E.

We have, for the second course, difference of latitude, 55.4 N., departure 23 W.

We have, for the third course, difference of latitude 42.1 S., departure 211.8 W.

On the whole, then, the ship has made 128.9 miles of north latitude, and 42.1 miles of south latitude. The place reached is therefore on a parallel of latitude 86.8 miles to the north of the parallel left; that is, in latitude 35° 49′.2 S.

The departure is, in the same way, found to be 161.3 miles W.; and the middle latitude is 36° 32′.6. With these data, and the formula of Art. 126, we find the difference of longitude to be 201 miles, or 3° 21′ W. Hence the longitude reached is 22° 3′ W.

With the difference of latitude 86.8 miles, and the departure 161.3 miles, we find the course to be N. 61° 43′ W., and the distance 183.2 miles. The ship has reached the same point that it would have reached, if it had sailed directly on a course N. 61° 43′ W., for a distance of 183.2 miles.

137. A ship leaves Cape Cod (Ex. 125), and sails S.E. by S. 114 miles, N. by E. 94 miles, W.N.W. 42 miles. Solve as in Art. 139.

138. A ship leaves Cape of Good Hope (latitude 34° 22′ S., longitude 18° 30′ E.), and sails N.W. 126 miles, N. by E. 84 miles, W.S.W. 217 miles. Solve as in Ex. 136.

EXAMINATION PAPERS.*

PLANE TRIGONOMETRY.

I.

(*Harvard College, Admission. June,* 1881. *Time,* 1½ *hours.*)

1. Define a logarithm. What is the logarithm of ½ in the system of which 16 is the base? Find the logarithm of 25 in the same system.

2. Compute the value of $\sqrt[5]{\frac{(0.012)^3 \times 0.27}{(64)^2 \times 0.00651}}$ by logarithms.

3. Find the functions of 127° 10′ from your trigonometric tables.

4. Prove the formula

$$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4 \sin^2 \frac{A-B}{2}.$$

5. Two sides of a triangle are 243 feet and 188 feet, and the angle opposite the second side is 42° 20′. Solve the triangle completely.

6. A pine tree growing on the side of a mountain, which is inclined to the horizontal at an angle of 20°, is broken by the wind but not severed at a distance of 40 feet from the ground. The top falls toward the foot of the mountain, and strikes the ground 50 feet from the base of the tree; find the height of the tree.

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* NOTE. In these papers, as in many text-books, the Greek letters α (*alpha*), β (*bayta*), γ (*gamma*), δ (*delta*), θ (*thayta*), ϕ (*phee*), are occasionally used to denote angles.

II.

(*Harvard College, Admission. June,* 1882. *Time,* 1½ *hours.*)

1. Explain the reason of the rule for finding the *characteristic* (or integral part) of the logarithm of a number.

Show that (according to this rule) the *mantissa* (or fractional part) is always *positive.*

In what cases is the logarithm, as a whole, positive, and in what cases negative?

Thus, state clearly the value of the logarithm of 36,270; of 0.003627. What decimal must be added to the latter logarithm to produce the logarithm of 0.01?

2. Find the time required to increase a sum of money a hundred fold, at *ten* per cent per annum, compound interest, payable yearly.

3. Find the formulas for the trigonometric functions of $90° + a$.

4. Find by the tables the logarithms of the trigonometric functions of 290° 38′ (marking the signs).

5. An observer from a ship saw two headlands. The first bore E.N.E. (*i.e.* 67° 30′ from N. towards E.), and the second N.W. by N. (*i.e.* 33° 45′ from N. towards W.). After he had sailed 16.25 miles N. by W. (*i.e.* 11° 15′ from N. towards W.), the first headland bore due E., and the second N.W. by W. (*i.e.* 56° 15′ from N. towards W.). Find the direction and distance of the second headland from the first.

6. Prove the formulas:

$$\frac{\cos a - \cos\beta}{\cos a + \cos\beta} = -\tan\tfrac{1}{2}(a+\beta)\tan\tfrac{1}{2}(a-\beta),$$

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}.$$

III.

(*Harvard College, Freshman Examination. April,* 1879. *Time,* 3 *hours.*)

1. Prove the relations between the sine, cosine, and tangent of $90° + \phi$ and the functions of ϕ. Draw a figure for the case where ϕ is obtuse, and show that the proof still holds good.

Confirm your results by means of the formulas for the sine and cosine of the sum of the two angles.

2. Deduce formulas for the sine, cosine, and tangent of 2α and $\frac{1}{2}\alpha$, in terms of functions of α.

3. Prove the formula:

$$\cos(\alpha+\beta)\sin\beta - \cos(\alpha+\gamma)\sin\gamma =$$
$$\sin(\alpha+\beta)\cos\beta - \sin(\alpha+\gamma)\cos\gamma.$$

4. Prove that in any triangle

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

5. Deduce the formulas for the tangents of the half angles of a triangle, in terms of the sides.

6. Solve the triangles:

$$C = 35°, \qquad a = 500, \qquad c = 250,$$
$$B = 22° \, 22', \qquad a = 67.06, \qquad b = 60.03.$$

7. The Delta measures 241 yds. on Cambridge St. and 115 yds. on Quincy St., and the angle between these streets is 88° 52′. Find the other angles of the Delta.

8. Find the area of the Delta.

9. A person travelling east in a railroad train observes a tower situated south of a station A, and on the same horizontal plane with the railroad. At the station B he finds that the distance of the tower is 2 miles; and at C, 3 miles from B, its distance is 4 miles. Find the distance of the tower from A.

IV.

(*Harvard College, Freshman Examination. April*, 1880. *Time*, 3 *hours*.)

1. Deduce the formula $\sin(\alpha+\beta) = \ldots\ldots$, drawing the figure for the case in which α is in the second quadrant and $\alpha+\beta$ in the third.

2. Deduce the formulas for $\cos 2\alpha$, $\sin\frac{1}{2}\alpha$, and $\cos\frac{1}{2}\alpha$, in terms of functions of α.

3. Prove the theorem of the sines.

4. From the fundamental formulas deduce the formula

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{a+b}{a-b}.$$

5. Prove that

$$\frac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = 1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha.$$

6. In a triangle $B = 4°13.4'$ and $a = 2001$, give all the solutions in the following cases:

(1) $b = 150$,
(2) $b = 200$,
(3) $b = 2001$.

7. A, B, and C are the corners of a triangular field. A is 40 ft. W. of B and 400 ft. S.W. of C. What is the area of the field? What is the length of the fence which encloses it?

8. From two corners of the Delta, A and B, lines which make angles of $19°52'$ and $57°32'$ respectively with the side AB meet directly under Memorial Hall tower. The length of AB is 345.1 ft., and the angle of elevation of the tower at A is $32°26'$. Find the height of the tower, and its angle of elevation at B.

V.

(*Harvard College, Freshman Examination. April,* 1881. *Time,* 3 *hours.*)

1. Deduce the formulas which connect the functions of $(90^\circ + \phi)$ and ϕ.

2. Prove the fundamental formula for $\cos(\alpha + \beta)$.

3. From the formula just found, obtain three values for $\cos 2\alpha$.

4. Find the values of $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$, in terms of $\cos\alpha$.

5. Prove the formula

$$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}.$$

6. Solve the following triangles:

$b = 2434$, $c = 1881$, $\mathrm{C} = 42^\circ\, 22'$.

$a = 0.00543$, $c = 0.07003$, $\mathrm{A} = 4^\circ\, 27'$.

7. The sides of a triangle are 715, 541, and 368; find one angle and the area.

8. The height of Memorial Hall tower is 190 feet. From its top the angles of depression of the corners of the Delta which lie on Cambridge St. are 57° 44′ and 16° 59′, and the angle subtended by the line joining these corners is 99° 30′. Find the length of the Delta on Cambridge St.

VI.

(*Harvard College, Freshman Examination. April,* 1882. *Time,* 3 *hours.*)

1. Obtain the formulas which connect the sine, cosine, and tangent of $(180^\circ + \phi)$ with the functions of ϕ.

2. Assuming the formulas for the sine and cosine of the sum of two angles, prove that

(1) $\tan(\alpha + \beta) =$

(2) $\sin\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1 - \cos\alpha)}$.

3. Find all the values of x, between 0° and 360°, which will satisfy the equations

$$(1)\quad \tan x = 2 \sin 2x,$$
$$(2)\quad (\sin x + \cos x)^2 = 2 \sin 2x.$$

4. The length of each side of a regular dodecagon is 24 feet; find the radius of the inscribed circle and the area of the polygon.

5. In a certain triangle, $a = 20$, $B = 3° 24'$, $C = 85° 31'$. Find c by aid of the table containing the values of S and T.

6. Prove the Theorem of Sines, and solve the triangles

$$(1)\quad b = 468, \qquad c = 327, \qquad C = 34° 15';$$
$$(2)\quad a = 0.003641, \quad c = 0.08091, \quad A = 5° 20'.$$

7. Given $a^2 = b^2 + c^2 - 2bc \cos A$; obtain the formula

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

The sides of a triangle are $a = 2408$, $b = 2028$, $c = 1884$; find the angle A.

8. Hingham is 12 miles south-east of Boston; Quincy is $5\frac{3}{4}$ miles west of Hingham. How far is Quincy from Boston?

9. A person ascending Memorial Hall tower stops to rest at a window, and notices that the angle of elevation of the vane on Appleton Chapel is 3° 34'. When he reaches the top of the tower, 190 feet above the ground, he finds that the height of the Chapel subtends an angle of 13° 9'. The horizontal distance between the two towers being 492 feet, find the height of Appleton Chapel and the distance of the window above the ground.

VII.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 7, 1876. *Time,* 2½ *hours.*)

1.* Assuming that the angle subtended at the centre of any circle by an arc equal to its radius is a constant angle, show that any angle may be expressed by the fraction $\frac{\text{arc}}{\text{radius}}$, the constant angle being taken as the unit.

Find the length of the arc subtended by an angle of 60° in a circle whose radius is 3 feet.

2. The sine of a certain angle is ¼; find the other trigonometrical ratios of the angle.

3. Trace the change in sign and magnitude in the tangent of an angle, as the angle increases from 0° to 360°.

4. Find, by a geometrical construction, the cosine of 60° and of 45°, and deduce the value of cos 3360° and cos 2565°.

5. Prove the formulas:

(1) $\sin(A-B) = \sin A \cos B - \cos A \sin B$,

(2) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$,

(3) $\dfrac{\sin 2A + \sin 4A}{\cos 2A + \cos 4A} = \tan 3A$.

6. Express the cosine of half an angle in terms of the sine of the angle, and explain the double sign.

Employ the formula to find the value of cos 75°, having given $\sin 150° = \frac{1}{2}$.

7. If A, B, C be the angles of a triangle, and a, b, c the sides respectively opposite to them, show that

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}},$$

where s = one-half the sum of the sides.

8. Find the greatest angle in a triangle whose sides are 7 feet, 8 feet, and 9 feet.

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* For aid in solving this and similar questions, see Wentworth & Hill's Tables, pages xi and xii.

VIII.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 7, 1877. *Time,* $2\frac{1}{2}$ *hours.*)

1. Define the cosine, cotangent, and cosecant of an angle, and prove that these ratios remain unchanged so long as the angle is the same.

Find the value of these three ratios for an angle of 45°.

2. Prove the formulas:

$$(1)\quad \sin A = \sqrt{1-\cos^2 A},$$

$$(2)\quad \cos A = \frac{1}{\sqrt{1+\tan^2 A}}.$$

If $\sec A = \sqrt{2}$, find $\tan A$. *Ans.* 1.

3. Prove that

$$\sin(90^\circ + A) = \cos A, \text{ and } \cos(90^\circ + A) = -\sin A.$$

Hence show that $\cos(180^\circ + A) = -\cos A$.

4. Show that $\cos^2 A \tan^2 A + \sin^2 A \cot^2 A = 1$.

5. Prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$. Hence show that

$$\frac{\cos(A+B+C)}{\sin A \sin B \sin C} = \cot A \cot B \cot C - \cot A - \cot B - \cot C.$$

6. Given that $\sin\frac{1}{2}A = \frac{3}{5}$, find the value of $\tan A$.

Ans. $\frac{24}{7}$.

7. Prove that the sides of any plane triangle are proportional to the sines of the angles opposite to these sides.

If $2s =$ the sum of the three sides (a, b, c) of a triangle, and if A be the angle opposite to the side a, prove that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

8. Prove that in any plane triangle

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\tfrac{1}{2}C.$$

9. If the side a and the angles A and B of a triangle be known, prove that the side b may be found by means of the formula,

$$\log b = \log a + \log \sin B - \log \sin A.$$

Find b, having given that $a = 1000$ yards, $A = 50°$, $B = 64°$.
Ans. 1173.29 yards.

10. The minute-hand of a clock is 3 feet 6 inches in length; find how far its point will move in a quarter of an hour, it being assumed that $\pi = \frac{22}{7}$. *Ans.* 5 feet 6 inches.

IX.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1878. *Time,* 2½ *hours.*)

1. Define sine, cotangent; and prove that $\sin^2 A + \cos^2 A = 1$. Express the other trigonometrical ratios in terms of the cosine.

2. What is meant by the circular measure of an angle? How is the number of degrees in an angle found from its circular measure? How many degrees are in the unit of circular measure?

3. Prove that

$$(1)\quad \sin(180° + A) = -\sin A,$$

$$(2)\quad \tan(90° + A) = -\cot A.$$

What is the use of these equations?

4. Find the general form of all the angles whose sine is the same as $\sin \theta$.

Write down the sines of all the angles which are multiples of 30° and less than 360°.

5. Prove the following relations:

(1) $\cos(A-B)=\cos A\cos B+\sin A\sin B$,

(2) $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$,

(3) $\tan^2 A=\dfrac{1-\cos 2A}{1+\cos 2A}$.

6. Find $\cos 30°$, $\tan 45°$, $\sin 15°$.

7. If $\tan A+\sec A=2$, prove that $\sin A=\frac{3}{5}$ when A is less than 90°.

If $\sin A=\frac{4}{5}$, prove that $\tan A+\sec A=3$ when A is less than 90°.

8. Prove that $\cos 3A=4\cos^3 A-3\cos A$, and find $\tan 3A$ in terms of $\tan A$.

9. In a triangle A, B, C, whose sides are respectively a, b, c, prove that

(1) $\sin\frac{1}{2}A=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$,

(2) $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

10. Solve a triangle, having given two sides and the angle opposite one of them.

Find A, B, b, having given $a=25$, $c=24$, $C=65°\,59'$.

X.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1879. *Time,* 2½ *hours.*)

1. Define 1°. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle $A°$ in circular measure.

2. Define the sine, secant, and cotangent of an angle, and express any two of these ratios in terms of the third.

Find the trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

3. Prove that

(1) $\cos(180^\circ + A) = \cos(180^\circ - A)$,

(2) $\tan\ (90^\circ + A) = \cot(180^\circ - A)$.

4. Express the cosine of the difference of two angles in terms of the sines and cosines of these angles.

Prove that

$$*\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

5. Prove the formulas:

(1) $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$,

(2) $\sin\frac{1}{2}x + \cos\frac{1}{2}x = \pm\sqrt{1+\sin x}$,

(3) $\sin x(2\cos x - 1) = 2\sin\frac{1}{2}x\ \cos\frac{3}{2}x$.

6. Trace the changes in sign and magnitude of

$$\frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta},$$

as θ changes from 0 to 2π.

7. Express the cosine of any angle of a triangle in terms of the sides of the triangle.

If the angle opposite the side a be 60°, and if b, c be the remaining sides of the triangle, prove that

$$(a+b+c)(b+c-a) = 3bc.$$

8. Solve a triangle, having given the three sides.

Given $A = 36^\circ$, $B = 72^\circ$, and $a = 1$; solve the triangle.

9. The sides of a triangle are 2, 3, 4; find the least angle.

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* $\tan^{-1}x$ = arc whose tangent is x.

XI.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1880. *Time,* 2½ *hours.*)

1. Express in degrees, minutes, etc., (i.) the angle whose circular measure is $\frac{1}{20}\pi$; (ii.) the angle whose circular measure is 5.

If the angle subtended at the centre of a circle by the side of a regular pentagon be the unit of angular measurement, by what number is a right angle represented?

2. Find, by geometrical constructions, the cosine of 45° and the sine of 120°.

Prove that

$$(\sin 30^\circ + \cos 30^\circ)(\sin 120^\circ + \cos 120^\circ) = \sin 30^\circ.$$

3. If $\csc A = 9$, find $\cot A$ and $\sec A$.

4. Prove that

$$\cos(180^\circ + A) = -\cos A.$$

Find the value of (i.) $\cot 840^\circ$; (ii.) $\sec 3\pi$.

5. Assuming the formula for the sine of the sum of two angles in terms of the sines and cosines of the separate angles, find (i.) $\sin 75^\circ$; (ii.) $\sin 3A$ in terms of $\sin A$.

6. Prove the formulas:

(1) $\cos^2(A - B) - \sin^2(A + B) = \cos 2A \cos 2B$,

(2) $1 + \tan x \tan \frac{1}{2}x = \sec x$.

7. Prove that

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B),$$

and express $\dfrac{\cos 5\theta + \cos\theta}{\cos 5\theta - \cos\theta}$ as a single term.

8. Solve the equations:

(1) $5 \tan^2 x + \sec^2 x = 7$,

(2) $\cos 5\theta + \cos 3\theta = \sqrt{2} \cos 4\theta$.

9. Prove that in any triangle $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.
Obtain the formula for $\tan \frac{1}{2} A$ in terms of the sides.

10. Find an expression for the area of a triangle in terms of its sides. The lengths of the sides of a triangle are 3 feet, 5 feet, and 6 feet; what is its area?

11. Given that

$$\sin 38^\circ 25' = 0.6213757, \quad \sin 38^\circ 26' = 0.6216036;$$

find the angle whose sine is 0.6215000.

XII.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1881. *Time,* 2½ *hours.*)

1. Define the unit of circular measure. The ratio of the circumference of a circle to its diameter being 3.14159, find the circular measure of an angle of 126°.

2. Define the tangent, cotangent, and cosecant of an angle. Find the tangent and cotangent of an angle whose cosecant is 1.25.

3. Trace the changes in sign and magnitude of $\sin A$ as A changes from 90° to 270°.

4. Prove the following:

$$(1)\quad \tan(\pi + A) = \tan A,$$

$$(2)\quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

$$(3)\quad \frac{\sin A + \sin B}{\cos A - \cos B} = \cot \frac{B - A}{2}.$$

5. Determine the value of $\cos 18°$, and prove that

$$\cos 36° = \cos 60° + \cos 72°.$$

6. Show that for certain values of the angles

$$2 \cos \tfrac{1}{2} A = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$

Is this formula true for values of A lying between 200° and 220°? and if not, how must it be modified?

7. Prove that in any triangle, with the usual notation,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}},$$

and that the area is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Show, also, that

$$\sin^2 A = \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C.$$

8. When one side of a triangle and the two adjacent angles are given, show how to solve the triangle.

Find the greatest side of the triangle, of which one side is 2183 feet, and the adjacent angles are 78° 14′ and 71° 24′.

SPHERICAL TRIGONOMETRY.

CHAPTER V.

THE RIGHT SPHERICAL TRIANGLE.

§ 45. INTRODUCTION.

The object of *Spherical Trigonometry* is to show how spherical triangles are solved. To *solve* a spherical triangle is to compute any three of its parts when the other three parts are given.

The sides of a spherical triangle are arcs of great circles. They are measured in degrees, minutes, and seconds, and therefore by the plane angles formed by radii of the sphere drawn to the vertices of the triangle. Hence, their measures are independent of the length of the radius, which may be assumed to have any convenient numerical value; as, for example, unity.

The angles of the triangle are measured by the angles made by the planes of the sides. Each angle is also measured by the number of degrees in the arc of a great circle, described from the vertex of the angle as a pole, and included between its sides.

The sides may have any values from 0° to 360°; but in this work only sides that are less than 180° will be considered. The angles may have any values from 0° to 180°.

If any two parts of a spherical triangle are either both less than 90° or both greater than 90°, they are said to be **alike in kind**; but if one part is less than 90°, and the other part greater than 90°, they are said to be **unlike in kind.**

Spherical triangles are said to be isosceles, equilateral, equiangular, right, and oblique, under the same conditions as plane triangles. A *right* spherical triangle, however, may have one, two, or three right angles.

When a spherical triangle has one or more of its sides equal to a quadrant, it is called a **quadrantal triangle.**

It is shown in Solid Geometry, that in every spherical triangle

I. *The sum of the sides is less than* 360°.

II. *The sum of the angles is greater than* 180°.

III. *If, from the vertices as poles, arcs of great circles are described, another spherical triangle is formed so related to the first triangle that the sides of each triangle are supplements of the angles opposite to them in the other triangle.*

Two such triangles are said to be *polar* with respect to each other.

Let A, B, C (Fig. 35) denote the angles of one triangle; a, b, c the sides opposite these angles respectively; and let A', B', C' and a', b', c' denote the corresponding sides and angles of the polar triangle. Then the above theorem gives the six following equations:

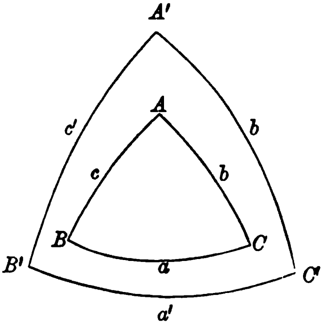

Fig. 35.

$$A + a' = 180°,$$
$$B + b' = 180°,$$
$$C + c' = 180°,$$
$$A' + a = 180°,$$
$$B' + b = 180°,$$
$$C' + c = 180°$$

EXERCISE XIX.

1. The angles of a triangle are 70°, 80°, and 100°; find the sides of the polar triangle.

2. The sides of a triangle are 40°, 90°, and 125°; find the angles of the polar triangle.

3. Prove that the polar of a quadrantal triangle is a right triangle.

4. Prove that, if a triangle have three right angles, the sides of the triangle are quadrants.

5. Prove that, if a triangle have two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the number of degrees in the opposite side.

6. How can the sides of a spherical triangle be found in units of length, when the length of the radius of the sphere is known?

7. Find the lengths of the sides of the triangle in Example 2, if the radius of the sphere is 4 feet.

§ 46. Formulas Relating to Right Triangles.

As is evident from § 45, Examples 4 and 5, the only kind of right triangle requiring further investigation is that which contains *only one* right angle.

Let ABC (Fig. 36) be such a right triangle, and let A, B, C denote the angles of the triangle; a, b, c, respectively, the opposite sides.

Let C be the right angle, and for the present suppose that each of the other parts is less than 90°.

Let planes be passed through the sides, intersecting in the radii OA, OB, and OC; and for the sake of simplicity let the radius of the sphere be taken equal to 1.

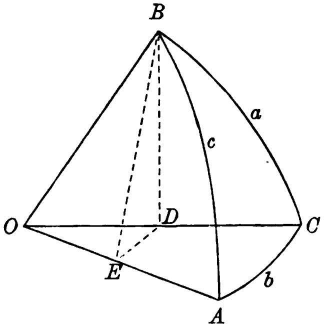

Fig. 36.

Also, let a plane perpendicular to OA be passed through B, cutting OA at E and OC at D. Join BE, BD, and DE.

BE and DE are each $\perp OA$ (Geom. § 454); therefore $\angle BED = A$. The plane BDE is $\perp$ the plane AOC (Geom. § 472); hence BD, which is the intersection of the planes BDE and BOC, is $\perp$ the plane AOC (Geom. § 475), therefore $\perp$ OC and DE. Now

$$\cos c = OE = OD \times \cos b,$$

and $OD = \cos a$.

$$\therefore \mathbf{\cos c = \cos a \cos b}. \qquad [37]$$

$$\sin a = BD = BE \times \sin A,$$

and $BE = \sin c$.

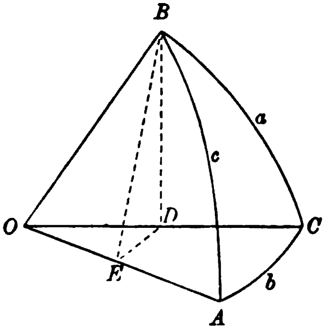

Fig. 36 (bis).

$$\begin{array}{ll} \text{Therefore,} & \mathbf{\sin a = \sin c \sin A} \\ \text{changing letters,} & \mathbf{\sin b = \sin c \sin B} \end{array} \Bigg\}. \qquad [38]$$

$$\begin{array}{lr} \text{Again,} & DE = BE \times \cos A, \\ \text{and also} & DE = OD \times \sin b; \\ \text{hence,} & BE \times \cos A = OD \times \sin b; \\ \text{that is,} & \sin c \cos A = \cos a \sin b; \\ \text{whence} & \cos A = \cos a \sin b \csc c.^{*} \end{array}$$

By substituting in this formula the value of $\cos a$, obtained from [37], we obtain

$$\begin{array}{ll} & \mathbf{\cos A = \tan b \cot c} \\ \text{In like manner,} & \mathbf{\cos B = \tan a \cot c} \end{array} \Bigg\}. \qquad [39]$$

And by substituting in the same formula the value of $\sin b$, as given in [38], we obtain

$$\begin{array}{ll} & \mathbf{\cos A = \cos a \sin B} \\ \text{In like manner,} & \mathbf{\cos B = \cos b \sin A} \end{array} \Bigg\}. \qquad [40]$$

$$\begin{array}{ll} \text{Also,} & BD = DE \times \tan A, \\ & BD = \sin a, \\ & DE = \cos a \sin b; \\ \text{therefore,} & \sin a = \cos a \sin b \tan A; \end{array}$$

---

* To avoid fractions, $\csc c$ is written in place of its equal, $\frac{1}{\sin c}$.

$$\left.\begin{array}{ll}\text{whence,} & \mathbf{\sin b = \tan a \cot A} \\ \text{and, similarly,} & \mathbf{\sin a = \tan b \cot B}\end{array}\right\}. \qquad [41]$$

If in [37] we substitute for $\cos a$ and $\cos b$ their values from [40], we obtain

$$\mathbf{\cos c = \cot A \cot B.} \qquad [42]$$

In deducing these formulas, it has been assumed that all the parts of the triangle, except the right angle, are less than 90°. But the formulas also hold true when this hypothesis is not fulfilled.

Let one of the legs a be greater than 90°, and construct a figure for this case (Fig. 37) in the same manner as Fig. 36.

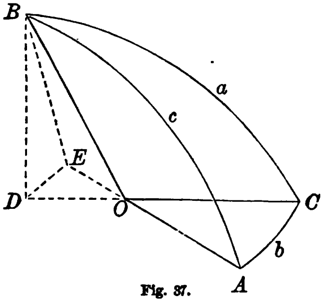

Fig. 37.

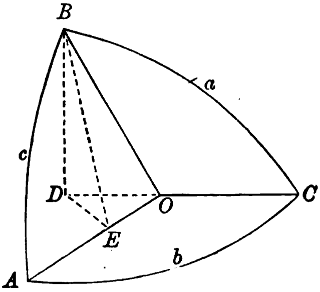

Fig. 38.

The auxiliary plane BDE will now cut both CO and AO produced beyond the centre O; and we have

$$\cos(180 - c) = OE = OD \times \cos b = \cos(180 - a) \cos b,$$

a result which reduces to [37] if we substitute $-\cos c$ in place of $\cos(180 - c)$ and $-\cos a$ in place of $\cos(180 - a)$. Likewise, the other formulas, [38]–[42], hold true in this case.

Again, suppose that both the legs a and b are greater than 90°. In this case the plane BDE (Fig. 38) will cut CO produced beyond O, and AO between A and O; and we have

$$\begin{aligned}\cos c = OE &= OD \times \cos(180 - b) \\ &= \cos(180 - a) \cos(180 - b) \\ &= \cos a \cos b,\end{aligned}$$

a result agreeing with [37]. And the remaining formulas may be easily shown to hold true.

Like results follow in all cases; in other words, Formulas [37]–[42] are universally true.

EXERCISE XX.

1. Prove, by aid of Formula [37], that the hypotenuse of a right triangle is *less than* or *greater than* 90°, according as the two legs are *alike* or *unlike* in kind.

2. Prove, by aid of Formula [40], that in a right spherical triangle each leg and the opposite angle are always alike in kind.

3. What inferences may be drawn respecting the values of the other parts: (i.) if $c = 90^\circ$; (ii.) if $a = 90^\circ$; (iii.) if $c = 90^\circ$ and $a = 90^\circ$; (iv.) if $a = 90^\circ$ and $b = 90^\circ$?

Deduce from [37]–[42] the following formulas:

4. $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c - a) \tan \frac{1}{2}(c + a)$.

5. $\tan^2 (45^\circ - \frac{1}{2} A) = \tan \frac{1}{2}(c - a) \cot \frac{1}{2}(c + a)$.

6. $\tan^2 \frac{1}{2} B = \sin (c - a) \csc (c + a)$.

7. $\tan^2 \frac{1}{2} c = -\cos (A + B) \sec (A - B)$.

8. $\tan^2 \frac{1}{2} a = \tan [\frac{1}{2}(A + B) - 45^\circ] \tan [\frac{1}{2}(A - B) + 45^\circ]$.

§ 47. NAPIER'S RULES.

The formulas deduced in § 46 express the relations between five parts of a right triangle, — the three sides and the two oblique angles. All these relations may be shown to follow from two very useful Rules, devised by Baron Napier, the inventor of Logarithms.

For this purpose the right angle (not entering the formulas) is left out of account, and instead of the hypotenuse and the two oblique angles, their respective *complements* are employed; so that the five parts considered by the Rules are: a, b, $co. c$, $co. A$, $co. B$. Any one of these parts may be called a *middle* part; and then the two parts immediately adjacent are called *adjacent* parts, and the other two are called *opposite* parts.

Rule I. *The sine of the middle part is equal to the product of the* **tan***gents of the* **ad***jacent parts.*

Rule II. *The sine of the middle part is equal to the product of the* **cos***ines of the* **op***posite parts.*

These Rules are easily remembered by the expressions, **tan. ad.** and **cos. op.**

The correctness of these Rules may be shown by taking each of the five parts as middle part, and comparing the resulting equations with the equations contained in Formulas [37]–[42].

For example, let *co. c* be taken as middle part, then *co. A* and *co. B* are the adjacent parts, and *a* and *b* the opposite parts,—as is very plainly seen in Fig. 39, in which the order of the parts is shown by arranging them around the circumference of a circle. Then, by Napier's Rules:

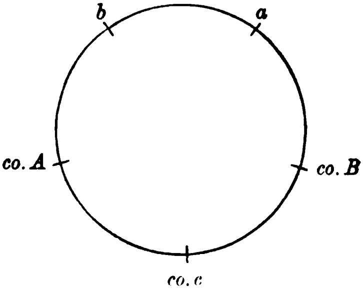

Fig. 39.

$$\sin(co.\,c) = \tan(co.\,A)\,\tan(co.\,B),$$

or

$$\cos c = \cot A\ \cot B;$$

$$\sin(co.\,c) = \cos a\ \cos b,$$

or

$$\cos c = \cos a\ \cos b;$$

results which agree with Formulas [37] and [42] respectively.

Exercise XXI.

1. Show that Napier's Rules lead to the equations contained in Formulas [38], [39], [40], and [41].

2. What will Napier's Rules become, if we take as the five parts of the triangle, the hypotenuse, the two oblique angles, and the *complements* of the two legs?

§ 48. SOLUTION OF RIGHT TRIANGLES.

By means of Formulas [37]–[42] we can solve a right triangle in all possible cases. In every case two parts besides the right angle must be given.

CASE I. *Given the two legs a and b.*
The solution is contained in Formulas [37] and [41]; viz:

$$\begin{aligned} \cos c &= \cos a \cos b, \\ \tan A &= \tan a \csc b, \\ \tan B &= \tan b \csc a. \end{aligned}$$

For example, let $a = 27° \, 28' \, 36''$, $b = 51° \, 12' \, 8''$; then the solution by logarithms is as follows:

$$\begin{array}{l} \log\cos a = 9.94802 \\ \underline{\log\cos b = 9.79697} \\ \log\cos c = 9.74499 \\ \qquad\quad c = 56° \, 13' \, 40'' \end{array}$$

$$\begin{array}{l|l} \log\tan a \;\, = 9.71604 & \log\tan b \;\, = 10.09477 \\ \underline{\log\csc b \;\, = 0.10826} & \underline{\log\csc a \;\, = \;\; 0.33593} \\ \log\tan A = 9.82430 & \log\tan B = 10.43070 \\ \qquad\quad A = 33° \, 42' \, 50'' & \qquad\quad B = 69° \, 38' \, 54'' \end{array}$$

CASE II. *Given the hypotenuse c and the leg a.*
From Formulas [37], [38], and [39] we obtain

$$\begin{aligned} \cos b &= \cos c \sec a, \\ \sin A &= \sin a \csc c, \\ \cos B &= \tan a \cot c. \end{aligned}$$

Although two angles in general correspond to $\sin A$, one acute the other obtuse, yet in this case the indetermination is removed by the fact that A and a must be alike in kind (see Exercise XX., Example 2).

CASE III. *Given the leg a and the opposite angle A.*

By means of Formulas [38], [40], and [41], we find, that

$$\begin{aligned}\sin c &= \sin a \csc A,\\ \sin b &= \tan a \cot A,\\ \sin B &= \sec a \cos A;\end{aligned}$$

or, from [37] and [39],

$$\begin{aligned}\cos b &= \cos c \sec a,\\ \cos B &= \tan a \cot c.\end{aligned}$$

When c has been computed, b and B are determined by these values of their cosines; but, since c must be found from its sine, c may have in general two values which are supplements of one another. This case, therefore, really admits of two solutions.

CASE IV. *Given the leg a and the adjacent angle B.*

Formulas [39], [40], and [41] give

$$\begin{aligned}\tan c &= \tan a \sec B,\\ \tan b &= \sin a \tan B,\\ \cos A &= \cos a \sin B.\end{aligned}$$

CASE V. *Given the hypotenuse c and the oblique angle A.* From Formulas [38], [39], and [42] it follows that

$$\begin{aligned}\sin a &= \sin c \sin A,\\ \tan b &= \tan c \cos A,\\ \cot B &= \cos c \tan A.\end{aligned}$$

Here a is determined by $\sin a$, since a and A must be alike in kind (see Exercise XX., Example 2).

CASE VI. *Given the two oblique angles A and B.*

By means of Formulas [40] and [42] we obtain

$$\begin{aligned}\cos c &= \cot A \cot B,\\ \cos a &= \cos A \csc B,\\ \cos b &= \cos B \csc A.\end{aligned}$$

NOTE 1. In Case I. (a and b given) the formula for computing c fails to give accurate results when c is very near 0° or 180°; in this case it may be found with greater accuracy by first computing B, and then computing c, as in Case IV.

NOTE 2. In Case II. (c and a given), if b is very near 0° or 180°, it may be computed more accurately by means of the derived formula

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c + a) \tan \tfrac{1}{2}(c - a).$$

And if A is so near 90° that it cannot be found accurately in the Tables it may be computed from the derived formula

$$\tan^2(45^\circ - \tfrac{1}{2} A) = \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a).$$

In like manner, when B cannot be accurately found from its cosine we may make use of the formula

$$\tan^2 \tfrac{1}{2} B = \sin(c - a) \csc(c + a).$$

NOTE 3. In Case III. (a and A given), when the formulas for the required parts do not give accurate results, we may employ the derived formulas

$$\begin{aligned}
\tan^2(45^\circ - \tfrac{1}{2} c) &= \tan \tfrac{1}{2}(A - a) \cot \tfrac{1}{2}(A + a),\\
\tan^2(45^\circ - \tfrac{1}{2} b) &= \sin(A - a) \csc(A + a),\\
\tan^2(45^\circ - \tfrac{1}{2} B) &= \tan \tfrac{1}{2}(A - a) \tan \tfrac{1}{2}(A + a).
\end{aligned}$$

NOTE 4. In Case IV. (a and B given), if A is near 0° or 180°, it may be more accurately found by first computing b and then finding A.

NOTE 5. In Case V. (c and A given), if a is near 90°, it may be found by first computing b, and then computing a by means of Formula [41].

NOTE 6. In Case VI. (A and B given), for unfavorable values of the sides greater accuracy may be obtained by means of the derived formulas

$$\begin{aligned}
\tan^2 \tfrac{1}{2} c &= -\cos(A + B) \sec(A - B),\\
\tan^2 \tfrac{1}{2} a &= \tan[\tfrac{1}{2}(A + B) - 45^\circ] \tan[45^\circ + \tfrac{1}{2}(A - B)],\\
\tan^2 \tfrac{1}{2} b &= \tan[\tfrac{1}{2}(A + B) - 45^\circ] \tan[45^\circ - \tfrac{1}{2}(A - B)].
\end{aligned}$$

NOTE 7. In Cases I., IV., and V., the solution is always possible; in Case II., in order that the solution should be possible, it is necessary and sufficient that $\sin a < \sin c$; in Cases III. and VI., the conditions of possibility follow obviously from the equations employed; in Case III., it is also necessary that a and A should be alike in kind.

Note 8. It is easy to trace analogies between the formulas for solving right spherical triangles and those for solving right plane triangles. The former, in fact, become identical with the latter if we suppose the radius of the sphere to be infinite in length; in which case the cosines of the sides become each equal to 1, and the ratios of the sines of the sides and of the tangents of the sides must be taken as equal to the ratios of the sides themselves.

If the formula required for any case is not remembered, it is always easy to find it by means of Napier's Rules. In applying these Rules we must choose for the middle part that one of the three parts considered — the two given and the one required — which will make the other two either adjacent parts or opposite parts.

For example: *given a and B; solve the triangle.*

First, represent the parts as in Fig. 40, and to prevent mistakes mark each of the given parts with a cross. To find b, take a as the middle part; then b and $co.\,B$ are adjacent parts; and by Rule I.,

$$\sin a = \tan b \cot B;$$

whence, $\tan b = \sin a \tan B$.

To find c, take $co.\,B$ as middle part; then a and $co.\,c$ are adjacent parts; and by Rule I.,

$$\cos B = \tan a \cot c;$$

whence, $$\tan c = \tan a \sec B.$$

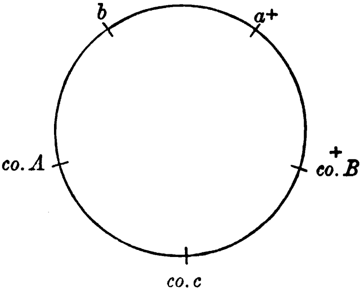

Fig. 40.

To find A, take $co.\,A$ as middle part; then a and $co.\,B$ are the opposite parts; and by Rule II.,

$$\cos A = \cos a \sin B.$$

In like manner, every case of a right spherical triangle may be solved.

EXERCISE XXII.

Solve the following right triangles, taking for the given parts in each case those printed in columns I. and II.:

	I.	II.	III.	IV.	V.
	a	*b*	*c*	*A*	*B*
1	36° 27′	43° 32′ 31″	54° 20′	46° 59′ 43.2″	57°59′19.3″
2	86° 40′	32° 40′	87° 11′ 39.8″	88° 11′ 57.8″	32°42′38.7″
3	50°	36° 54′ 49″	59° 4′ 25.7″	63° 15′ 13.1″	44°26′21.6″
4	120° 10′	150° 59′ 44″	63° 55′ 43.2″	105° 44′ 21.25″	147°19′47.14″
	c	*a*	*b*	*A*	*B*
5	55° 9′ 32″	22° 15′ 7″	51° 53′	27° 28′ 25.7″	73° 27′ 11.16″
6	23° 49′ 51″	14° 16′ 35″	19° 17′	37° 36′ 49.4″	54° 49′ 23.3″
7	44° 33′ 17″	32° 9′ 17″	32° 41′	49° 20′ 16.4″	50° 19′ 16″
8	97° 13′ 4″	132° 14′ 12″	79° 13′ 38.2″	131° 43′ 50″	81° 58′ 53.3″
	a	*A*	*c*	*b*	*B*
9	77° 21′ 50″	83° 56′ 40″	78° 53′ 20″	28° 14′ 31.3″	28° 49′ 57.4″
			101° 6′ 40″	151° 45′ 28.7″	151° 10′ 2.6″
10	77° 21′ 50″	40° 40′ 40″	impossible;	why?	
	a	*B*	*c*	*b*	*A*
11	92° 47′ 32″	50° 2′ 1″	91° 47′ 40″	50°	92° 8′ 23″
12	2° 0′ 55″	12° 40′	2° 3′ 55.7″	0° 27′ 10.2″	77° 20′ 28.4″
13	20° 20′ 20″	38° 10′ 10″	25° 14′ 38.2″	15° 16′ 50.4″	54° 35′ 16.7″
14	54° 30′	35° 30′	59° 51′ 20.8″	30° 8′ 39.2″	70° 17′ 35″
	c	*A*	*a*	*b*	*B*
15	69° 25′ 11″	54° 54′ 42″	50°	56° 50′ 49.3″	63° 25′ 4″
16	112° 48′	56° 11′ 56″	50°	127° 4′ 30″	120° 3′ 50″
17	46° 40′ 12″	37° 46′ 9″	26° 27′ 24″	39° 57′ 41.5″	62° 0′ 4″
18	118° 40′ 1″	128° 0′ 4″	136° 15′ 32.3″	48° 23′ 38.4″	58° 27′ 4.3″
	A	*B*	*a*	*b*	*c*
19	63° 15′ 12″	135° 33′ 39″	49° 59′ 56″	143° 5′ 12″	120° 55′ 34.3″
20	116° 43′ 12″	116° 31′ 25″	120° 10′ 3″	119° 59′ 46″	75° 26′ 58″
21	46° 59′ 42″	57° 59′ 17″	36° 27′	43° 32′ 30″.	54° 20′
22	90°	88° 24′ 35″	90°	88° 24′ 35″	90°

23. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

24. Solve the quadrantal triangle whose sides are:

$$a = 174° 12' 49.1'', \quad b = 94° 8' 20'', \quad c = 90°.$$

25. Solve the quadrantal triangle in which

$$c = 90°, \quad A = 110° 47' 50'', \quad B = 135° 35' 34.5''.$$

26. Given in a spherical triangle A, C, and $c = 90°$; solve the triangle.

27. Given $A = 60°$, $C = 90°$, and $c = 90°$; solve the triangle.

28. Given in a right spherical triangle, $A = 42° 24' 9''$, $B = 9° 4' 11''$; solve the triangle.

29. In a right triangle, given $a = 119° 11'$, $B = 126° 54'$; solve the triangle.

30. In a right triangle, given $c = 50°$, $b = 44° 18' 39''$; solve the triangle.

31. In a right triangle, given $A = 156° 20' 30''$, $a = 65° 15' 45''$; solve the triangle.

32. If the legs a and b of a right spherical triangle are equal, prove that $\cos a = \cot A = \sqrt{\cos c}$.

33. In a right triangle prove that $\cos^2 A \times \sin^2 c = \sin(c - a) \sin(c + a)$.

34. In a right triangle prove that $\tan a \cos c = \sin b \cot B$.

35. In a right triangle prove that

$$\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

36. In a right triangle prove that

$$\sin(b + c) = 2 \cos^2 \tfrac{1}{2} A \cos b \sin c.$$

37. In a right triangle prove that

$$\sin(c - b) = 2 \sin^2 \tfrac{1}{2} A \cos b \sin c.$$

38. If, in a right triangle, p denote the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the legs a and b respectively, prove that (i.) $\tan^2 a = \tan c \tan m$, (ii.) $\sin^2 p = \tan m \tan n$.

§ 49. SOLUTION OF THE ISOSCELES SPHERICAL TRIANGLE.

If an arc of a great circle is passed through the vertex of an isosceles spherical triangle and the middle point of its base, the triangle will be divided into two symmetrical right triangles. In this way the solution of an isosceles spherical triangle may be reduced to that of a right spherical triangle.

In a similar manner the solution of a regular spherical polygon may be reduced to that of a right spherical triangle. Arcs of great circles, passed through the centre of the polygon and its vertices, divide it into a series of equal isosceles triangles; and each one of these may be divided into two equal right triangles.

EXERCISE XXIII.

1. In an isosceles spherical triangle, given the base b and the side a; find A the angle at the base, B the angle at the vertex, and h the altitude.

2. In an equilateral spherical triangle, given the side a; find the angle A.

3. Given the side a of a regular spherical polygon of n sides; find the angle A of the polygon, the distance R from the centre of the polygon to one of its vertices, and the distance r from the centre to the middle point of one of its sides.

4. Compute the dihedral angles made by the faces of the five regular polyhedrons.

5. A spherical square is a spherical quadrilateral which has equal sides and equal angles. Its two diagonals divide it into four equal right triangles. Find the angle A of the square, having given the side a.

CHAPTER VI.

THE OBLIQUE SPHERICAL TRIANGLE.

§ 51. FUNDAMENTAL FORMULAS.

Let ABC (Fig. 41) be an oblique spherical triangle, a, b, c its three sides, A, B, C the angles opposite to them, respectively.

Through C draw an arc CD of a great circle, perpendicular to the side AB, meeting AB at D. For brevity let $CD = p$, $AD = m$, $BD = n$, $\angle ACD = x$, $\angle BCD = y$.

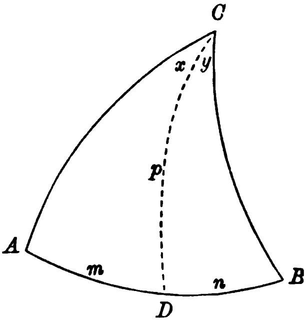

Fig. 41.

1. By § 46 [38], in the right triangles BDC and ADC,

$$\sin p = \sin a \sin B,$$
and
$$\sin p = \sin b \sin A.$$

Therefore, similarly, and

$$\left.\begin{aligned} \sin a \sin B &= \sin b \sin A \\ \sin a \sin C &= \sin c \sin A \\ \sin b \sin C &= \sin c \sin B \end{aligned}\right\}. \qquad [43]$$

These equations may of course also be written in the form of proportions; as, for example,

$$\sin a : \sin b = \sin A : \sin B.$$

In Fig. 41 the arc of the great circle CD cuts the side AB within the triangle. In case it cut AB produced without the triangle, $\sin(180° - A)$, $\sin(180° - B)$, or $\sin(180° - C)$, would be employed in the above proof instead of $\sin A$, $\sin B$, or $\sin C$. These sines, however, are equal to $\sin A$, $\sin B$, and $\sin C$, respectively, so that the Formulas [43] hold true in all cases.

2. In the right triangle BDC, by § 46 [37],

$$\cos a = \cos p \cos n = \cos p \cos(c-m),$$

or (§ 32) $\cos a = \cos p \cos c \cos m + \cos p \sin c \sin m.$

Now, $\cos p \cos m = \cos b;$ (§ 46 [37])

whence, $\cos p = \cos b \sec m,$

and $\cos p \sin m = \cos b \tan m$

$= \cos b \tan b \cos A$ (§ 46 [39])

$= \sin b \cos A.$

Substituting these values of $\cos p \cos m$ and $\cos p \sin m$ in the value of $\cos a$, we obtain

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A\\ \text{and similarly,}\quad \cos b &= \cos a \cos c + \sin a \sin c \cos B\\ \cos c &= \cos a \cos b + \sin a \sin b \cos C\end{aligned}\right\}. \qquad [44]$$

3. In the right triangle ADC, by § 40 [41],

$$\cos A = \cos p \sin x = \cos p \sin(C-y),$$

or (§ 32) $\cos A = \cos p \sin C \cos y - \cos p \cos C \sin y.$

Now, $\cos p \sin y = \cos B;$ (§ 46 [40])

whence, $\cos p = \cos B \csc y,$

and $\cos p \cos y = \cos B \cot y$

$= \cos B \tan B \cos a$ (§ 46 [42])

$= \sin B \cos a.$

Substituting these values of $\cos p \sin y$ and $\cos p \cos y$ in the value of $\cos A$, we obtain

$$\left.\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a\\ \text{and similarly,}\quad \cos B &= -\cos A \cos C + \sin A \sin C \cos b\\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c\end{aligned}\right\}. \qquad [45]$$

Formulas [44] and [45] are also universally true; for the same equations are obtained when the arc CD cuts the side AB without the triangle.

EXERCISE XXIV.

1. What do Formulas [43] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $a = 90°$? if $A = B = 90°$? if $a = b = 90°$?

2. What does the first of [44] become if $A = 0°$? if $A = 90°$? if $A = 180°$?

3. From Formulas [44] deduce Formulas [45], by means of the relations between polar triangles (§ 45).

§ 52. FORMULAS FOR THE HALF ANGLES AND SIDES.

From the first equation of [44],

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

whence,

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c},$$

$$1 + \cos A = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c};$$

or, by § 34 [16] and [17], and § 35 [23],

$$\sin^2 \tfrac{1}{2} A = \sin \tfrac{1}{2} (a + b - c) \sin \tfrac{1}{2} (a - b + c) \csc b \csc c,$$

$$\cos^2 \tfrac{1}{2} A = \sin \tfrac{1}{2} (a + b + c) \sin \tfrac{1}{2} (b + c - a) \csc b \csc c.$$

Now, let $\tfrac{1}{2}(a + b + c) = s;$

whence,

$$\tfrac{1}{2}(b + c - a) = s - a,$$

$$\tfrac{1}{2}(a - b + c) = s - b,$$

$$\tfrac{1}{2}(a + b - c) = s - c.$$

Then, by substitution and extraction of the square root,

$$\left.\begin{aligned}\sin\tfrac{1}{2}A &= \sqrt{\sin(s-b)\sin(s-c)\csc b\csc c}\\ \cos\tfrac{1}{2}A &= \sqrt{\sin s\sin(s-a)\csc b\csc c}\\ \tan\tfrac{1}{2}A &= \sqrt{\csc s\csc(s-a)\sin(s-b)\sin(s-c)}\end{aligned}\right\}.\ [46]$$

In like manner it may be shown that

$$\begin{aligned}\sin\tfrac{1}{2}B &= \sqrt{\sin(s-a)\sin(s-c)\csc a\csc c},\\ \cos\tfrac{1}{2}B &= \sqrt{\sin s\sin(s-b)\csc a\csc c},\\ \tan\tfrac{1}{2}B &= \sqrt{\csc s\csc(s-b)\sin(s-a)\sin(s-c)}.\end{aligned}$$

$$\begin{aligned}\sin\tfrac{1}{2}C &= \sqrt{\sin(s-a)\sin(s-b)\csc a\csc b},\\ \cos\tfrac{1}{2}C &= \sqrt{\sin s\sin(s-c)\csc a\csc b},\\ \tan\tfrac{1}{2}C &= \sqrt{\csc s\csc(s-c)\sin(s-a)\sin(s-b)}.\end{aligned}$$

Again, from the first equation of [45],

$$\cos a = \frac{\cos B\cos C + \cos A}{\sin B\sin C};$$

whence,

$$1-\cos a = \frac{\sin B\sin C - \cos B\cos C - \cos A}{\sin B\sin C},$$

$$1+\cos a = \frac{\sin B\sin C + \cos B\cos C + \cos A}{\sin B\sin C}.$$

If we place $\tfrac{1}{2}(A+B+C)=S$, and proceed in the same manner as before, we obtain the following results:

$$\left.\begin{aligned}\sin\tfrac{1}{2}a &= \sqrt{-\cos S\cos(S-A)\csc B\csc C}\\ \cos\tfrac{1}{2}a &= \sqrt{\cos(S-B)\cos(S-C)\csc B\csc C}\\ \tan\tfrac{1}{2}a &= \sqrt{-\cos S\cos(S-A)\sec(S-B)\sec(S-C)}\end{aligned}\right\}.[47]$$

And, in like manner,

$$\begin{aligned}\mathbf{\sin \tfrac{1}{2} b} &= \mathbf{\sqrt{-\cos S \cos(S-B) \csc A \csc C}},\\ \mathbf{\cos \tfrac{1}{2} b} &= \mathbf{\sqrt{\cos(S-A) \cos(S-C) \csc A \csc C}},\\ \mathbf{\tan \tfrac{1}{2} b} &= \mathbf{\sqrt{-\cos S \cos(S-B) \sec(S-A) \sec(S-C)}}.\end{aligned}$$

$$\begin{aligned}\mathbf{\sin \tfrac{1}{2} c} &= \mathbf{\sqrt{-\cos S \cos(S-C) \csc A \csc B}},\\ \mathbf{\cos \tfrac{1}{2} c} &= \mathbf{\sqrt{\cos(S-A) \cos(S-B) \csc A \csc B}},\\ \mathbf{\tan \tfrac{1}{2} c} &= \mathbf{\sqrt{-\cos S \cos(S-C) \sec(S-A) \sec(S-B)}}.\end{aligned}$$

§ 53. GAUSS'S EQUATIONS AND NAPIER'S ANALOGIES.

By § 31 [5],

$$\cos\tfrac{1}{2}(A+B) = \cos\tfrac{1}{2}A \cos\tfrac{1}{2}B - \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B;$$

or, by substituting for $\cos\frac{1}{2}A$, $\cos\frac{1}{2}B$, $\sin\frac{1}{2}A$, $\sin\frac{1}{2}B$, their values given in § 52, and reducing,

$$\begin{aligned}\cos\tfrac{1}{2}(A+B) &= \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \times \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}}\\ &\quad - \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}} \times \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin a \sin c}}\\ &= \frac{\sin s - \sin(s-c)}{\sin c} \times \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}}.\end{aligned}$$

This value, by applying §§ 33 [12], 35 [21], and observing that the quantity under the radical is equal to $\sin\frac{1}{2}C$, becomes

$$\cos\tfrac{1}{2}(A+B) = \frac{2 \sin\frac{1}{2}c \cos(s-\frac{1}{2}c)}{2 \sin\frac{1}{2}c \cos\frac{1}{2}c} \sin\tfrac{1}{2}C,$$

which, by cancelling common factors, multiplying by $\cos\frac{1}{2}c$, and observing that $s - \frac{1}{2}c = \frac{1}{2}(a+b)$, reduces to the form

$$\cos\tfrac{1}{2}(A+B) \cos\tfrac{1}{2}c = \cos\tfrac{1}{2}(a+b) \sin\tfrac{1}{2}C.$$

By proceeding in like manner with the values of

$$\sin\tfrac{1}{2}(A+B),\quad \cos\tfrac{1}{2}(A-B),\quad \text{and } \sin\tfrac{1}{2}(A-B),$$

three analogous equations are obtained.

The four equations,

$$\left.\begin{aligned}
\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C\\
\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C\\
\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C\\
\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C
\end{aligned}\right\} \qquad [48]$$

are called Gauss's Equations.

By dividing the second of Gauss's Equations by the first, the fourth by the third, the third by the first, and the fourth by the second, we obtain

$$\left.\begin{aligned}
\tan\tfrac{1}{2}(A+B) &= \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
\tan\tfrac{1}{2}(A-B) &= \frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
\tan\tfrac{1}{2}(a+b) &= \frac{\cos\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c\\
\tan\tfrac{1}{2}(a-b) &= \frac{\sin\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c
\end{aligned}\right\}. \qquad [49]$$

These equations are called Napier's Analogies.

In the first equation the factors $\cos\tfrac{1}{2}(a-b)$ and $\cot\tfrac{1}{2}C$ are always positive; therefore, $\tan\tfrac{1}{2}(A+B)$ and $\cos\tfrac{1}{2}(a+b)$ must always have like signs. Hence, if $a+b<180°$, and therefore $\cos\tfrac{1}{2}(a+b)>0$, then, also, $\tan\tfrac{1}{2}(A+B)>0$, and therefore $A+B<180°$. Similarly, it follows that if $a+b>180°$, then, also, $A+B>180°$. If $a+b=180°$, and therefore $\cos\tfrac{1}{2}(a+b)=0$, then $\tan\tfrac{1}{2}(A+B)=\infty$; whence $\tfrac{1}{2}(A+B)=90°$, and $A+B=180°$.

Conversely, it may be shown from the third equation, that $a+b$ is less than, greater than, or equal to 180°, according as $A+B$ is less than, greater than, or equal to 180°.

§ 54. Case I.

Given two sides, a and b, and the included angle C.

The angles A and B may be found by the first two of Napier's Analogies; viz.:

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

After A and B have been found, the side c may be found by [43]; but it is better to use for this purpose Gauss's Equations, because they involve functions of the same angles that occur in working Napier's Analogies. Any one of the equations may be used; for example, from the first we have

$$\cos \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(A+B)} \sin \tfrac{1}{2} C.$$

Example. $a = 73°\,58'\,54''$, $b = 38°\,45'\,0''$, $C = 46°\,33'\,41''$, therefore, $\tfrac{1}{2}(a-b) = 17°\,36'\,57''$, $\tfrac{1}{2}(a+b) = 56°\,21'\,57''$, $\tfrac{1}{2}C = 23°\,16'\,50.5''$

$\log\cos \frac{1}{2}(a-b)$	$= 9.97914$	$\log\sin \frac{1}{2}(a-b)$	$= 9.48092$
$\log\sec \frac{1}{2}(a+b)$	$= 0.25658$	$\log\csc \frac{1}{2}(a+b)$	$= 0.07956$
$\log\cot \frac{1}{2}C$	$= 0.36626$	$\log\cot \frac{1}{2}C$	$= 0.36626$
$\log\tan \frac{1}{2}(A+B)$	$= 0.60198$	$\log\tan \frac{1}{2}(A-B)$	$= 9.92674$
$\log\sec \frac{1}{2}(A+B)$	$= 0.61547$	$\frac{1}{2}(A+B)$	$= 75°\,57'\,40.7''$
$\log\cos \frac{1}{2}(a+b)$	$= 9.74342$	$\frac{1}{2}(A-B)$	$= 40°\,11'\,25.6''$
$\log\sin \frac{1}{2}C$	$= 9.59686$	A	$= 116°\,9'\,6.3''$
$\log\cos \frac{1}{2}c$	$= 9.95575$	B	$= 35°\,46'\,15.1''$
$\frac{1}{2}c$	$= 25°\,25'\,40''$	c	$= 50°\,51'\,20''$

If the side c only is desired, it may be found from [44], without previously computing A and B. But the Formulas [44] are not adapted to logarithmic work. Instead of changing them to forms suitable for logarithms, we may use the following method, which leads to the same results, and has the advantage that, in applying it, nothing has to be remembered except Napier's Rules:

Make the triangle (Fig. 42), as in § 51, equal to the sum (or the difference) of two right triangles. For this purpose, through B (or A, but *not* C) draw an arc of a great circle perpendicular to AC, cutting AC at D. Let $BD = p$, $CD = m$, $AD = n$; and mark with crosses the given parts.

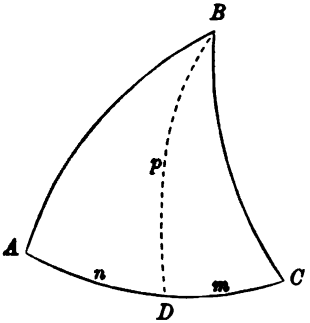

Fig. 42.

By Rule I.,

$$\cos C = \tan m \cot a,$$

whence $\tan m = \tan a \cos C$.

By Rule II.,

$$\cos a = \cos m \cos p, \text{ whence } \cos p = \cos a \sec m.$$
$$\cos c = \cos n \cos p, \text{ whence } \cos p = \cos c \sec n.$$

Therefore, $\cos c \sec n = \cos a \sec m$;

or, since $n = b - m$, $\cos c = \cos a \sec m \cos(b - m)$.

It is evident that c may be computed, with the aid of logarithms, from the two equations

$$\tan m = \tan a \cos C,$$
$$\cos c = \cos a \sec m \cos(b - m).$$

EXAMPLE. Given $a = 97°30'20''$, $b = 55°12'10''$, $C = 39°58'$; find c.

$$\begin{aligned} \log \tan a &= 0.88025(n) \\ \log \cos C &= 9.88447 \\ \hline \log \tan m &= 0.76472(n) \\ m &= 99°45'14'' \\ b - m &= -44°33'\ 4'' \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.11602(n) \\ \log \cos(b-m) &= 9.85286 \\ \log \sec m &= 0.77103(n) \\ \hline \log \cos c &= 9.73991 \\ c &= 56°40'20'' \end{aligned}$$

EXERCISE XXV.

1. Write formulas for finding, by Napier's Rules, the side a when b, c, and A are given, and for finding the side b when a, c, and B are given.

2. Given $a = 88^\circ\,12'\,20''$, $b = 124^\circ\,7'\,17''$, $C = 50^\circ\,2'\,1''$; find $A = 63^\circ\,15'\,11''$, $B = 132^\circ\,17'\,59''$, $c = 59^\circ\,4'\,18''$.

3. Given $a = 120^\circ\,55'\,35''$, $b = 88^\circ\,12'\,20''$, $C = 47^\circ\,42'\,1''$; find $A = 129^\circ\,58'\,3''$, $B = 63^\circ\,15'\,9''$, $c = 55^\circ\,52'\,40''$.

4. Given $b = 63^\circ\,15'\,12''$, $c = 47^\circ\,42'\,1''$, $A = 59^\circ\,4'\,25''$; find $B = 88^\circ\,12'\,24''$, $C = 55^\circ\,52'\,42''$, $a = 50^\circ\,1'\,40''$.

5. Given $b = 69^\circ\,25'\,11''$, $c = 109^\circ\,46'\,19''$, $A = 54^\circ\,54'\,42''$; find $B = 56^\circ\,11'\,57''$, $C = 123^\circ\,21'\,12''$, $a = 67^\circ\,13'$.

§ 55. CASE II.

Given two angles, A and B, and the included side c.

The sides a and b may be found by the third or fourth of Napier's Analogies,

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c,$$

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c,$$

and then the angle C may be found by one of Gauss's equations; as, for instance, the second, which gives

$$\cos\tfrac{1}{2}C = \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(a-b)}\cos\tfrac{1}{2}c.$$

EXAMPLE. $A = 107^\circ\,47'\,7''$
$B = 38^\circ\,58'\,27''$
$c = 51^\circ\,41'\,14''$

$\log\cos\frac{1}{2}(A-B) = 9.91648$
$\log\sec\frac{1}{2}(A+B) = 0.54359$
$\log\tan\frac{1}{2}c = 9.68517$
$\log\tan\frac{1}{2}(a+b) = 0.14524$
$\log\sin\frac{1}{2}(A+B) = 9.98146$
$\log\sec\frac{1}{2}(a-b) = 0.01703$
$\log\cos\frac{1}{2}c = 9.95423$
$\log\cos\frac{1}{2}C = 9.95272$
$\frac{1}{2}C = 26^\circ\,14'\,52.5''$

$\frac{1}{2}(A-B) = 34^\circ\,24'\,20''$
$\frac{1}{2}(A+B) = 73^\circ\,22'\,47''$
$\frac{1}{2}c = 25^\circ\,50'\,37''$

$\log\sin\frac{1}{2}(A-B) = 9.75208$
$\log\csc\frac{1}{2}(A+B) = 0.01854$
$\log\tan\frac{1}{2}c = 9.68517$
$\log\tan\frac{1}{2}(a-b) = 9.45579$
$\frac{1}{2}(a+b) = 54^\circ\,24'\,24.4''$
$\frac{1}{2}(a-b) = 15^\circ\,56'\,25.6''$

$$\begin{cases} a = 70^\circ\,20'\,50'' \\ b = 38^\circ\,27'\,59'' \\ C = 52^\circ\,29'\,45'' \end{cases}$$

If the angle C alone is wanted, the best way is to decompose the triangle into two right triangles, and then apply Napier's Rules, as in Case I., when the side c alone is desired.

Let (Fig. 43) $\angle ABD = x$, $\angle CBD = y$, $BD = p$; then,

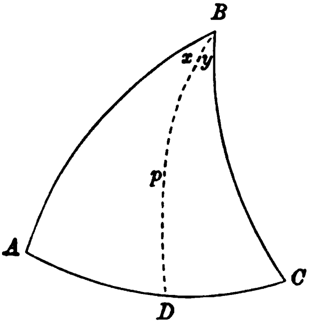

Fig. 43.

By Rule I.,

$$\cos c = \cot x \cot A,$$
whence $\cot x = \tan A \cos c$.

By Rule II.,

$$\cos A = \cos p \sin x,$$
whence $\cos p = \cos A \csc x$.

$$\cos C = \cos p \sin y,$$
whence $\cos p = \cos C \csc y$.

Therefore,

$$\cos C = \cos A \csc x \sin y = \cos A \csc x \sin (B - x).$$

It is clear that C may be computed from the equations

$$\cot x = \tan A \cos c,$$
$$\cos C = \cos A \csc x \sin (B - x).$$

Example. Given $A = 35° 46' 15''$, $B = 115° 9' 7''$, $c = 51° 2'$; find C.

$$\begin{array}{rl} \log \tan A & = 9.85760 \\ \log \cos c & = 9.79856 \\ \hline \log \cot x & = 9.65616 \\ x & = 65° 37' 35'' \\ B - x & = 49° 31' 32'' \end{array} \qquad \begin{array}{rl} \log \cos A & = 9.90992 \\ \log \sin (B - x) & = 9.88122 \\ \log \csc x & = 0.04055 \\ \hline \log \cos C & = 9.83099 \\ C & = 47° 20' 30'' \end{array}$$

Exercise XXVI.

1. What are the formulas for computing A when B, C, and a are given; and for computing B when A, C, and b are given?

2. Given $A = 26° 58' 46''$, $B = 39° 45' 10''$, $c = 154° 46' 48''$; find $a = 37° 14' 10''$, $b = 121° 28' 10''$, $C = 161° 22' 11''$.

3. Given $A = 128° 41' 49''$, $B = 107° 33' 20''$, $c = 124° 12' 31''$; find $a = 125° 41' 44''$, $b = 82° 47' 34''$, $C = 127° 22'$.

4. Given $B = 153° 17' 6''$, $C = 78° 43' 36''$, $a = 86° 15' 15''$; find $b = 152° 43' 51''$, $c = 88° 12' 21''$, $A = 78° 15' 48''$.

5. Given $A = 125° 41' 44''$, $C = 82° 47' 35''$, $b = 52° 37' 57''$; find $a = 128° 31' 46''$, $c = 107° 33' 20''$, $B = 55° 47' 40''$.

§ 56. CASE III.

Given two sides a and b, and the angle A opposite to one of them.

The angle B is found from [43], whence we have

$$\sin B = \sin A \sin b \csc a.$$

When B has been found, C and c may be found from the fourth and the second of Napier's Analogies, from which we obtain

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

The third and the first of Napier's Analogies may also be used for the same purpose.

NOTE 1. Since B is determined from its sine, the problem in general has two solutions; and, moreover, in case $\sin B > 1$, the problem is impossible. By geometric construction it may be shown, as in the corresponding case in Plane Trigonometry, under what conditions the problem really has two solutions, one solution, and no solution. But in practical applications a general knowledge of the shape of the triangle is known beforehand; so that it is easy to see, without special investigation, which solution (if any) corresponds to the circumstances of the question.

It can be shown that there are two solutions, when

$$A < 90°, \quad a + b < 180°, \quad \text{and } a < b,$$

or, when

$$A > 90°, \quad a + b > 180°, \quad \text{and } a > b.$$

Note 2. The side c or the angle C may be computed, without first finding B, by means of the formulas

$$\tan m = \cos A \tan b, \text{ and } \cos(c - m) = \cos a \times \sec b \times \cos m,$$
$$\cot x = \tan A \cos b, \text{ and } \cos(C - x) = \cot a \times \tan b \times \cos x.$$

These formulas may be obtained by resolution of the triangle into right triangles, and applying Napier's Rules; m is equal to that part of the side c included between the foot of the perpendicular from C and the vertex A, and x is equal to the corresponding portion of the angle C.

Example. Given $a = 57° 36'$, $b = 31° 12'$, $A = 104° 25' 30''$.

In this case $A > 90°$,
and $a + b < 180°$;
therefore, $A + B < 180°$;
hence, $B < 90°$,
and only one solution.

$$\begin{aligned} \log\sin A &= 9.98609 \\ \log\sin b &= 9.71435 \\ \log\csc a &= 0.07349 \\ \hline \log\sin B &= 9.77393 \\ B &= 36° 27' 20'' \end{aligned}$$

$$\begin{aligned} a + b &= 88° 50' \\ a - b &= 26° 26' \\ A + B &= 140° 51' 53'' \\ A - B &= 67° 59' \ 7'' \\ \log\sin \tfrac{1}{2}(A + B) &= 9.97416 \\ \log\csc \tfrac{1}{2}(A - B) &= 0.25252 \\ \log\tan \tfrac{1}{2}(a - b) &= 9.37080 \\ \hline \log\tan \tfrac{1}{2}c &= 9.59748 \\ \tfrac{1}{2}c &= 21° 35' 38'' \\ c &= 43° 11' 16'' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(a + b) &= 44° 25' \\ \tfrac{1}{2}(a - b) &= 13° 13' \\ \tfrac{1}{2}(A + B) &= 70° 26' 25'' \\ \tfrac{1}{2}(A - B) &= 33° 59' \ 5'' \\ \log\sin \tfrac{1}{2}(a + b) &= 9.84502 \\ \log\csc \tfrac{1}{2}(a - b) &= 0.64086 \\ \log\tan \tfrac{1}{2}(A - B) &= 9.82873 \\ \hline \log\cot \tfrac{1}{2}C &= 0.31461 \\ \tfrac{1}{2}C &= 25° 51' 15'' \\ C &= 51° 42' 30'' \end{aligned}$$

EXERCISE XXVII.

1. Given $a = 73° 49' 38''$, $b = 120° 53' 35''$, $A = 88° 52' 42''$; find $B = 116° 42' 30''$, $c = 120° 57' 27''$, $C = 116° 47' 4''$.

2. Given $a = 150° 57' 5''$, $b = 134° 15' 54''$, $A = 144° 22' 42''$; find $B_1 = 120° 47' 45''$, $c_1 = 55° 42' 8''$, $C_1 = 97° 42' 55.4''$;
$B_2 = 59° 12' 15''$, $c_2 = 23° 57' 17.4''$, $C_2 = 29° \ 8' 39''$.

3. Given $a = 79° 0' 54.5''$, $b = 82° 17' 4''$, $A = 82° 9' 25.8''$; find $B = 90°$, $c = 45° 12' 19''$, $C = 45° 44'$.

4. Given $a = 30° 52' 36.6''$, $b = 31° 9' 16''$, $A = 87° 34' 12''$; show that the triangle is impossible.

§ 57. CASE IV.

Given two angles A and B, and the side a opposite to one of them.

The side b is found from [43], whence

$$\sin b = \sin a \sin B \csc A.$$

The values of c and C may then be found by means of Napier's Analogies, the fourth and second of which give

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

NOTE 1. In this case, also, an unknown part is found from its sine; and it may be shown that, under certain conditions, the problem is impossible, or that it admits of two solutions. In practice, the ambiguity is usually removed by the circumstances of the question. If $\sin b > 1$ the problem of course is impossible; and it may be shown that there are two solutions, when

$a < 90°$, $A + B < 180°$, and $A < B$,

or $a > 90°$, $A + B > 180°$, and $A > B$.

NOTE 2. By proceeding as indicated in Case III., Note 2, formulas for computing c or C, independent of the side b, may be found; viz.:

$\tan m = \tan a \cos B$, and $\sin(c - m) = \cot A \tan B \sin m$,

$\cot x = \cos a \tan B$, and $\sin(C - x) = \cos A \sec B \sin x$.

In these formulas $m = BD$, $x = \angle BCD$, D being the foot of the perpendicular from the vertex C.

EXERCISE XXVIII.

1. Given $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$; find $b = 155° 5' 18''$, $c = 33° 1' 36''$, $C = 70° 20' 40''$.

2. Given $A = 113° 39' 21''$, $B = 123° 40' 18''$, $a = 65° 39' 46''$; find $b = 124° 7' 20''$, $c = 159° 53' 2''$, $C = 159° 43' 35''$.

3. Given $A = 100^\circ 2' 11.3''$, $B = 98^\circ 30' 28''$, $a = 95^\circ 20' 38.7''$; find $b = 90^\circ$, $c = 147^\circ 41' 43''$, $C = 148^\circ 5' 33''$.

4. Given $A = 24^\circ 33' 9''$, $B = 38^\circ 0' 12''$, $a = 65^\circ 20' 13''$; show that the triangle is impossible.

§ 58. CASE V.

Given the three sides, a, b, and c.

The angles are computed by means of Formulas [46], and the corresponding formulas for the angles B and C.

The formulas for the tangent are in general to be preferred.

If we multiply the equation

$$\tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s - a) \sin (s - b) \sin (s - c)}$$

by the equation $$1 = \frac{\sin (s - a)}{\sin (s - a)},$$

and put

$$\sqrt{\csc s \sin (s - a) \sin (s - b) \sin (s - c)} = \tan r,$$

and also make analogous changes in the equations for $\tan \frac{1}{2} B$ and $\tan \frac{1}{2} C$, we obtain

$$\begin{aligned} \tan \tfrac{1}{2} A &= \tan r \csc (s - a), \\ \tan \tfrac{1}{2} B &= \tan r \csc (s - b), \\ \tan \tfrac{1}{2} C &= \tan r \csc (s - c), \end{aligned}$$

which are the most convenient formulas to employ when all three angles have to be computed.

EXAMPLE 1.

$$\begin{aligned} a &= 50^\circ 54' 32'' \\ b &= 37^\circ 47' 18'' \\ c &= 74^\circ 51' 50'' \\ \hline 2s &= 163^\circ 33' 40'' \\ s &= 81^\circ 46' 50'' \\ s - a &= 30^\circ 52' 18'' \\ s - b &= 43^\circ 59' 32'' \\ s - c &= 6^\circ 55' \ 0'' \end{aligned}$$

$$\begin{aligned} \log \csc s &= 0.00448 \\ \log \csc (s - a) &= 0.28978 \\ \log \sin (s - b) &= 9.84171 \\ \log \sin (s - c) &= 9.08072 \\ & 2\overline{)19.21669} \\ \log \tan \tfrac{1}{2} A &= 9.60835 \\ \tfrac{1}{2} A &= 22^\circ \ 5' 20'' \\ A &= 44^\circ 10' 40'' \end{aligned}$$

Example 2.

$$\begin{aligned} a &= 124^\circ 12' 31'' \\ b &= 54^\circ 18' 16'' \\ c &= 97^\circ 12' 25'' \\ \hline 2s &= 275^\circ 43' 12'' \end{aligned}$$

$$\begin{aligned} s &= 137^\circ 51' 36'' \\ s-a &= 13^\circ 39' 5'' \\ s-b &= 83^\circ 33' 20'' \\ s-c &= 40^\circ 39' 11'' \end{aligned}$$

$$\begin{aligned} \log\sin(s-a) &= 9.37293 \\ \log\sin(s-b) &= 9.99725 \\ \log\sin(s-c) &= 9.81390 \\ \log\csc\, s &= 0.17331 \\ \hline \log\tan^2 r &= 9.35739 \\ \log\tan r &= 9.67870 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}A &= 0.30577 \\ \log\tan\tfrac{1}{2}B &= 9.68145 \\ \log\tan\tfrac{1}{2}C &= 9.86480 \\ \hline \tfrac{1}{2}A &= 63^\circ 41' 3.8'' \\ \tfrac{1}{2}B &= 25^\circ 39' 5.6'' \\ \tfrac{1}{2}C &= 36^\circ 13' 20.1'' \\ \hline A &= 127^\circ 22' 7'' \\ B &= 51^\circ 18' 11'' \\ C &= 72^\circ 26' 40'' \end{aligned}$$

Exercise XXIX.

1. Given $a = 120^\circ 55' 35''$, $b = 59^\circ 4' 25''$, $c = 106^\circ 10' 22''$; find $A = 116^\circ 44' 49''$, $B = 63^\circ 15' 14''$, $C = 91^\circ 7' 21''$.

2. Given $a = 50^\circ 12' 4''$, $b = 116^\circ 44' 48''$, $c = 129^\circ 11' 42''$; find $A = 59^\circ 4' 28''$, $B = 94^\circ 23' 12''$, $C = 120^\circ 4' 52''$.

3. Given $a = 131^\circ 35' 4''$, $b = 108^\circ 30' 14''$, $c = 84^\circ 46' 34''$; find $A = 132^\circ 14' 21''$, $B = 110^\circ 10' 40''$, $C = 99^\circ 42' 24''$.

4. Given $a = 20^\circ 16' 38''$, $b = 56^\circ 19' 40''$, $c = 66^\circ 20' 44''$; find $A = 20^\circ 9' 54''$, $B = 55^\circ 52' 31''$, $C = 114^\circ 20' 17''$.

§ 59. Case VI.

Given the three angles, A, B, and C.

The sides are computed by means of Formulas [47], and the corresponding formulas for the angles B and C. The formulas for the tangents are in general to be preferred.

If we multiply the equation

$$\tan\tfrac{1}{2}a = \sqrt{-\cos S\cos(S-A)\sec(S-B)\sec(S-C)}$$

by the equation $$1 = \frac{\sec(S-A)}{\sec(S-A)},$$
and put

$$\sqrt{-\cos S\sec(S-A)\sec(S-B)\sec(S-C)} = \tan R,$$

and also make analogous changes in the equations for $\tan \frac{1}{2} b$ and $\tan \frac{1}{2} c$, we obtain

$$\begin{aligned} \tan \tfrac{1}{2} a &= \tan R \cos(S-A), \\ \tan \tfrac{1}{2} b &= \tan R \cos(S-B), \\ \tan \tfrac{1}{2} c &= \tan R \cos(S-C), \end{aligned}$$

which are the most convenient formulas to use in case all three angles have to be computed.

Example I.

$$\begin{aligned} A &= 220^\circ \\ B &= 130^\circ \\ C &= 150^\circ \\ \hline 2S &= 500^\circ \\ S &= 250^\circ \\ S-A &= 30^\circ \\ S-B &= 120^\circ \\ S-C &= 100^\circ \end{aligned}$$

$$\begin{aligned} \log\cos S &= 9.53405 \; (n) \\ \log\cos(S-A) &= 9.93753 \\ \log\sec(S-B) &= 0.30103 \; (n) \\ \log\sec(S-C) &= 0.76033 \; (n) \\ & \quad 2\overline{)0.53294} \\ \log\tan\tfrac{1}{2} a &= 0.26647 \\ \tfrac{1}{2} a &= 61^\circ 34' \; 6'' \\ a &= 123^\circ \; 8' 12'' \end{aligned}$$

Note. Here the effect, as regards algebraic sign, of three negative factors, is cancelled by the negative sign belonging to the whole fraction.

Example II.

$$\begin{aligned} A &= 20^\circ \; 9' 56'' \\ B &= 55^\circ 52' 32'' \\ C &= 114^\circ 20' 14'' \\ \hline 2S &= 190^\circ 22' 42'' \end{aligned}$$

$$\begin{aligned} S &= 95^\circ 11' 21'' \\ S-A &= 75^\circ \; 1' 25'' \\ S-B &= 39^\circ 18' 49'' \\ S-C &= -19^\circ \; 8' 53'' \end{aligned}$$

$$\begin{aligned} \log\cos S &= 8.95638 \; (n) \\ \log\sec(S-A) &= 0.58768 \\ \log\sec(S-B) &= 0.11143 \\ \log\sec(S-C) &= 0.02472 \\ \hline \log\tan^2 R &= 9.68021 \\ \log\tan R &= 9.84010 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2} a &= 9.25242 \\ \log\tan\tfrac{1}{2} b &= 9.72867 \\ \log\tan\tfrac{1}{2} c &= 9.81538 \\ \hline \tfrac{1}{2} a &= 10^\circ \; 8' 18.9'' \\ \tfrac{1}{2} b &= 28^\circ \; 9' 50.4'' \\ \tfrac{1}{2} c &= 33^\circ 10' 21.3'' \\ \hline a &= 20^\circ 16' 38'' \\ b &= 56^\circ 19' 41'' \\ c &= 66^\circ 20' 43'' \end{aligned}$$

Exercise XXX.

1. Given $A = 130^\circ$, $B = 110^\circ$, $C = 80^\circ$;
find $a = 139^\circ 21' 22''$, $b = 126^\circ 57' 52''$, $c = 56^\circ 51' 48''$.

2. Given $A = 59^\circ 55' 10''$, $B = 85^\circ 36' 50''$, $C = 59^\circ 55' 10''$;
find $a = 128^\circ 42' 29''$, $b = 64^\circ 2' 47''$, $c = 128^\circ 42' 29''$.

3. Given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$; find $a = 104° 25' 9''$, $b = 53° 49' 25''$, $c = 97° 44' 24''$.

4. Given $A = 4° 23' 35''$, $B = 8° 28' 20''$, $C = 172° 17' 56''$; find $a = 31° 9' 11''$, $b = 84° 18' 23''$, $c = 115° 9' 56''$.

§ 60. AREA OF A SPHERICAL TRIANGLE.

I. *When the three angles A, B, C, are given.*

Let R = radius of sphere,
E = the spherical excess $= A + B + C - 180°$,
F = area of triangle;

then, by Solid Geometry,

$$F = \frac{E}{180°}\pi R^2. \qquad [50]$$

II. *When the three sides a, b, c, are given.*

A formula for computing the area is deduced as follows: From the first of [48],

$$\frac{\cos\frac{1}{2}(A+B)}{\cos(90° - \frac{1}{2}C)} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

whence, by the Theory of Proportions,

$$\frac{\cos\frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos\frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)} = \frac{\cos\frac{1}{2}(a+b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}. \qquad (a)$$

Now, in § 35, the division of [23] by [22] gives

$$\frac{\cos A - \cos B}{\cos A + \cos B} = -\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B), \qquad (b)$$

in which for A and B we may substitute any other two angular magnitudes, as for example, $\frac{1}{2}(A + B)$ and $(90 - \frac{1}{2}C)$, or $\frac{1}{2}(a + b)$ and $\frac{1}{2}c$.

If we use in place of A and B the values $\frac{1}{2}(A + B)$ and $(90° - \frac{1}{2}C)$, the first side of equation (b) becomes

$$\frac{\cos\frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos\frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)};$$

and the second side becomes

$$-\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B+90^\circ-\tfrac{1}{2}C)\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B-90^\circ+\tfrac{1}{2}C);$$

or,

$$-\tan\tfrac{1}{4}(A+B-C+180^\circ)\tan\tfrac{1}{4}(A+B+C-180^\circ).$$

If we remember that $E=A+B+C-180^\circ$, and observe that

$$\begin{aligned}\tan\tfrac{1}{4}(A+B-C+180^\circ)&=\tan\tfrac{1}{4}(360^\circ-2C+A+B+C-180^\circ)\\&=\tan\tfrac{1}{4}(360^\circ-2C+E)\\&=\tan[90^\circ-\tfrac{1}{4}(2C-E)]\\&=\cot\tfrac{1}{4}(2C-E),\end{aligned}$$

it will be evident that equation (b) may be written

$$\frac{\cos\frac{1}{2}(A+B)-\cos(90^\circ-\frac{1}{2}C)}{\cos\frac{1}{2}(A+B)+\cos(90^\circ-\frac{1}{2}C)}=-\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E.\qquad\text{(c)}$$

If we substitute, in equation (b), for A and B, the values $\frac{1}{2}(a+b)$ and $\frac{1}{2}c$, and also substitute s for $\frac{1}{2}(a+b+c)$ and $s-c$ for $\frac{1}{2}(a+b-c)$, equation (b) will become

$$\frac{\cos\frac{1}{2}(a+b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}=-\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c).\qquad\text{(d)}$$

Comparing (a), (c), and (d), we obtain

$$\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c).\qquad\text{(e)}$$

By beginning with the second equation of [48], and treating it in the same way, we obtain as the result,

$$\tan\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b).\qquad\text{(f)}$$

By taking the product of (e) and (f), we obtain the elegant formula,

$$\tan^2\tfrac{1}{4}\mathrm{E}=\tan\tfrac{1}{2}\mathrm{s}\tan\tfrac{1}{2}(\mathrm{s}-\mathrm{a})\tan\tfrac{1}{2}(\mathrm{s}-\mathrm{b})\tan\tfrac{1}{2}(\mathrm{s}-\mathrm{c}),\qquad[51]$$

which is known as l'Huilier's Formula.

By means of it E may be computed from the three sides, and then the area of the triangle may be found by [50].

III. *In all other cases*, the area may be found by first solving the triangle so far as to obtain the angles or the sides, whichever may be more convenient, and then applying [50] or [51].

EXAMPLE I.

$$\begin{aligned} A &= 102^\circ\,14'\,12'' \\ B &= 54^\circ\,32'\,24'' \\ C &= 89^\circ\,5'\,46'' \\ \hline &\ 245^\circ\,52'\,22'' \\ E &= 65^\circ\,52'\,22'' \\ &= 237142'' \\ 180^\circ &= 648000'' \end{aligned}$$

$$\begin{aligned} \log E &= 5.37501 \\ {}^*\log\frac{\pi}{648000} &= 4.68557-10 \\ \hline &\ 0.06058 \\ F &= 1.1497\,R^2 \end{aligned}$$

EXAMPLE II.

$$\begin{aligned} a &= 133^\circ\,26'\,19'' \\ b &= 64^\circ\,50'\,53'' \\ c &= 144^\circ\,13'\,45'' \\ \hline 2s &= 342^\circ\,30'\,57'' \\ \hline s &= 171^\circ\,15'\,28.5'' \\ s-a &= 37^\circ\,49'\,9.5'' \\ s-b &= 106^\circ\,24'\,35.5'' \\ s-c &= 27^\circ\,1'\,43.5'' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}s &= 85^\circ\,37'\,44'' \\ \tfrac{1}{2}(s-a) &= 18^\circ\,54'\,35'' \\ \tfrac{1}{2}(s-b) &= 53^\circ\,12'\,18'' \\ \tfrac{1}{2}(s-c) &= 13^\circ\,30'\,52'' \\ \hline \log\tan\tfrac{1}{2}s &= 1.11669 \\ \log\tan\tfrac{1}{2}(s-a) &= 9.53474 \\ \log\tan\tfrac{1}{2}(s-b) &= 0.12612 \\ \log\tan\tfrac{1}{2}(s-c) &= 9.38083 \\ \hline \log\tan^2\tfrac{1}{4}E &= 0.15838 \\ \log\tan\tfrac{1}{4}E &= 0.07919 \\ \tfrac{1}{4}E &= 50^\circ\,11'\,43'' \\ E &= 200^\circ\,46'\,52'' \end{aligned}$$

EXERCISE XXXI.

1. Given $A = 84^\circ\,20'\,19''$, $B = 27^\circ\,22'\,40''$, $C = 75^\circ\,33'$; find $E = 26159''$, $F = 0.12685\,R^2$.

2. Given $a = 69^\circ\,15'\,6''$, $b = 120^\circ\,42'\,47''$, $c = 159^\circ\,18'\,33''$; find $E = 216^\circ\,40'\,18''$.

3. Given $a = 33^\circ\,1'\,45''$, $b = 155^\circ\,5'\,18''$, $C = 110^\circ\,10'$; find $E = 133^\circ\,48'\,53''$.

4. Find the spherical excess of a triangle on the earth's surface (regarded as spherical), if each side of the triangle is equal to 1°.

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* See Wentworth & Hill's Tables, page 20.

CHAPTER VII.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

§ 61. PROBLEM.

To reduce an angle measured in space to the horizon.

Let O (Fig. 44) be the position of the observer on the ground, $AOB = h$ the angle measured in space (for example, the angle between the tops of two church spires), OA' and OB' the projections of the sides of the angle upon the horizontal plane HR, $AOA' = m$ and $BOB' = n$ the angles of inclination of OA and OB respectively to the horizon. Required the angle $A'OB' = x$ made by the projections on the horizon.

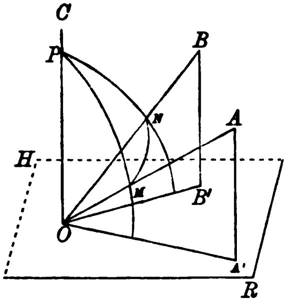


Fig. 44.

The planes of the angles of inclination AOA' and BOB' produced intersect in the line OC, which is perpendicular to the horizontal plane (Geom. § 475).

From O as a centre describe a sphere, and let its surface cut the edges of the trihedral angle $O = ABC$ in the points M, N, and P. In the spherical triangle MNP the three sides $MN = h$, $MP = 90 - m$, $NP = 90 - n$, are known, and the spherical angle P is equal to the required angle x.

From § 52 we obtain

$$\cos \tfrac{1}{2} x = \sqrt{\cos s \cos (s - h) \sec m \sec n},$$

where $\tfrac{1}{2}(m + n + h) = s$.

§ 62. PROBLEM.

To find the distance between two places on the earth's surface (regarded as spherical), given the latitudes of the places and the difference of their longitudes.

Let M and N (Fig. 45) be the places; then their distance MN is an arc of the great circle passing through the places. Let P be the pole, AB the equator. The arcs MR and NS are the latitudes of the places, and the arc RS, or the angle MPN, is the difference of their longitudes. Let $MR = a$, $NS = b$, $RS = l$; then in the spherical triangle MNP two sides, $MP = 90 - a$, $NP = 90 - b$, and the included angle $MPN = l$, are given, and we have (from § 54)

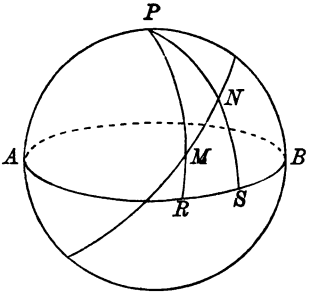


Fig. 45.

$$\tan m \;\; = \cot a \cos l,$$
$$\cos MN = \sin a \sec m \sin (b + m).$$

From these equations first find m, then the arc MN, and then reduce MN to geographical miles, of which there are 60 in each degree.

§ 63. THE CELESTIAL SPHERE.

The **Celestial Sphere** is an imaginary sphere of indefinite radius, upon the concave surface of which all the heavenly bodies appear to be situated.

The **Celestial Equator,** or **Equinoctial,** is the great circle in which the plane of the earth's equator produced intersects the surface of the celestial sphere.

The **Poles** of the equinoctial are the points where the earth's axis produced cuts the surface of the celestial sphere.

The **Celestial Meridian** of an observer is the great circle in which the plane of his terrestrial meridian produced meets the surface of the celestial sphere.

Hour Circles, or **Circles of Declination,** are great circles passing through the poles, and perpendicular to the equinoctial.

The **Horizon** of an observer is the great circle in which a plane tangent to the earth's surface, at the place where he is, meets the surface of the celestial sphere.

The **Zenith** of an observer is that pole of his horizon which is exactly above his head.

Vertical Circles are great circles passing through the zenith of an observer, and perpendicular to his horizon.

The vertical circle passing through the east and west points of the horizon is called the **Prime Vertical**; that passing through the north and south points coincides with the celestial meridian.

The **Ecliptic** is a great circle of the celestial sphere, apparently traversed by the sun in one year from west to east, in consequence of the motion of the earth around the sun.

The **Equinoxes** are the points where the ecliptic cuts the equinoctial. They are distinguished as the *Vernal* equinox and the *Autumnal* equinox; the sun in his annual journey passes through the former on March 21, and through the latter on September 21.

Circles of Latitude are great circles passing through the poles of the ecliptic, and perpendicular to the plane of the ecliptic.

The angle which the ecliptic makes with the equinoctial is called the **obliquity** of the ecliptic; it is equal to $23^\circ 27'$, nearly, and is often denoted by the letter e.

These definitions are illustrated in Figs. 46 and 47. In Fig. 46, $AVBU$ is the equinoctial, P and P' its poles, $NPZS$ the celestial meridian of an observer, $NESW$ his horizon, Z his zenith, M a star, PMP' the hour circle passing through the star, $ZMDZ'$ the vertical through the star.

In Fig. 47, *AVBU* represents the equinoctial, *EVFU* the ecliptic, *P* and *Q* their respective poles, *V* the vernal equinox, *U* the autumnal equinox, *M* a star, *PMR* the hour circle through the star, *QMT* the circle of latitude through the star, and $\angle TVR = e$.

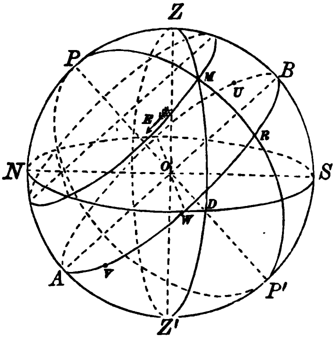


Fig. 46.

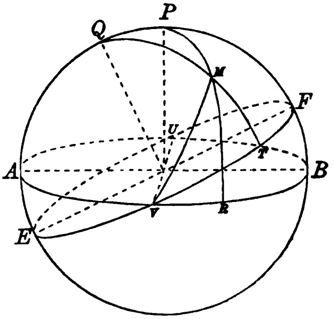


Fig. 47.

The earth's diurnal motion causes all the heavenly bodies to appear to rotate from east to west at the uniform rate of 15° per hour. If in Fig. 46 we conceive the observer placed at the centre *O*, and his zenith, horizon, and celestial meridian fixed in position, and all the heavenly bodies rotating around *PP'* as an axis from east to west at the rate of 15° per hour, we form a correct idea of the apparent diurnal motions of these bodies. When the sun or a star in its diurnal motion crosses the meridian, it is said to make a *transit* across the meridian; when it passes across the part *NWS* of the horizon, it is said to *set;* and when it passes across the part *NES*, it is said to *rise* (the effect of refraction being here neglected). Each star, as *M*, describes daily a small circle of the sphere parallel to the equinoctial, and called the **Diurnal Circle** of the star. The diurnal circle is the smaller the nearer the star is to the pole; and if there were stars at the poles *P* and *P'*, they would have no diurnal motion. To an observer north of the equator, the north pole P is *elevated* above the horizon (as shown in Fig. 46); to an observer south of the equator, the south pole P' is the elevated pole.

§ 64. SPHERICAL CO-ORDINATES.

Several systems of fixing the position of a star on the surface of the celestial sphere at any instant are in use. In each system a great circle and its pole are taken as standards of reference, and the position of the star is determined by means of two quantities called its *spherical co-ordinates.*

I. *If the horizon and the zenith* are chosen, the co-ordinates of the star are called its altitude and its azimuth.

The **Altitude** of a star is its angular distance, measured on a vertical circle, above the horizon. The complement of the altitude is called the **Zenith Distance.**

The **Azimuth** of a star is the angle at the zenith formed by the meridian of the observer and the vertical circle passing through the star, and is measured therefore by an arc of the horizon. It is usually reckoned from the north point of the horizon in north latitudes, and from the south point in south latitudes; and east or west according as the star is east or west of the meridian.

II. *If the equinoctial and its pole* are chosen, then the position of the star may be fixed by means of its declination and its hour angle.

The **Declination** of a star is its angular distance from the equinoctial, measured on an hour circle. The angular distance of the star, measured on the hour circle, *from the elevated pole* is called its **Polar Distance.**

The declination of a star, like the latitude of a place on the earth's surface, may be either north or south; but, in practical problems, while latitude is always to be considered positive, declination, if of a different name from the latitude, must be regarded as *negative.*

If the declination is negative, the polar distance is equal numerically to 90° + the declination.

The **Hour Angle** of a star is the angle at the pole formed by the meridian of the observer and the hour circle passing through the star. On account of the diurnal rotation, it is constantly changing at the rate of 15° per hour. Hour angles are reckoned from the celestial meridian, positive towards the west, and negative towards the east.

III. The *equinoctial and its pole* being still retained, we may employ as the co-ordinates of the star its declination and its right ascension.

The **Right Ascension** of a star is the arc of the equinoctial included between the vernal equinox and the point where the hour circle of the star cuts the equinoctial Right ascension is reckoned from the vernal equinox eastward from 0° to 360°.

IV. *The ecliptic and its pole* may be taken as the standards of reference. The co-ordinates of the star are then called its latitude and its longitude.

The **Latitude** of a star is its angular distance from the ecliptic measured on a circle of latitude.

The **Longitude** of a star is the arc of the ecliptic included between the vernal equinox and the point where the circle of latitude through the star cuts the ecliptic.

In problems it is useful to employ certain letters to denote these various co-ordinates. For a star M (Fig. 46), let

$l =$ latitude of the observer,
$h = DM$ = the altitude of the star,
$z = ZM$ = the zenith distance of the star,
$a = \angle PZM =$ the azimuth of the star,
$t = \angle ZPM =$ the hour angle of the star,
$d = RM$ = the declination of the star,
$p = PM$ = the polar distance of the star,
$r = VR$ = the right ascension of the star,
$u = MT$ (Fig. 47) = the latitude of the star,
$v = VT$ (Fig. 47) = the longitude of the star.

In many problems, a simple way of representing the magnitudes involved, is to project the sphere on the plane of the horizon, as shown in Fig. 48.

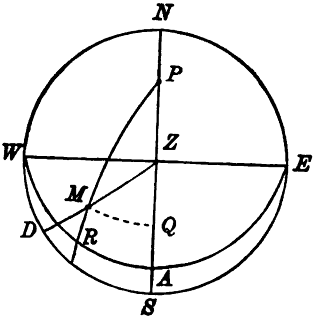


Fig. 48.

$NESW$ is the horizon, Z the zenith, NZS the meridian, WZE the prime vertical, WAE the equinoctial projected on the plane of the horizon, P the elevated pole, M a star, DM its altitude, ZM its zenith distance, $\angle PZM$ its azimuth, MR its declination, PM its polar distance, $\angle ZPM$ its hour angle.

§ 65. THE ASTRONOMICAL TRIANGLE.

The triangle ZPM (Figs. 46 and 48) is often called the *astronomical triangle*, on account of its importance in problems in Nautical Astronomy.

The side PZ is equal to the complement of the latitude of the observer. For (Fig. 46) the angle ZOB between the zenith of the observer and the celestial equator is obviously equal to his latitude, and the angle POZ is the complement of ZOB. The arc NP being the complement of PZ, it follows that *the altitude of the elevated pole is equal to the latitude of the place of observation.*

The triangle ZPM then (however much it may vary in shape for different positions of the star M), always contains the following five magnitudes:

$PZ =$ co-latitude of observer $= 90° - l$,
$ZM =$ zenith distance of star $= z$,
$PZM =$ azimuth of star $= a$,
$PM =$ polar distance of star $= p$,
$ZPM =$ hour angle of star $= t$.

A very simple relation exists between the hour angle of the sun and the local (apparent) time of day. Since the hourly rate at which the sun appears to move from east to west is 15°, and it is apparent noon when the sun is on the meridian of a place, it is evident that if hour angle $= 0°, 15°, -15°$, etc., time of day is noon, 1 o'clock P.M., 11 o'clock A.M., etc.

In general, if t denote the absolute value of the hour angle,

$$\text{time of day} = \frac{t}{15}\text{ P.M., or } 12 - \frac{t}{15}\text{ A.M.,}$$

according as the sun is west or east of the meridian.

§ 66. PROBLEM.

Given the latitude of the observer and the altitude and azimuth of a star, to find its declination and its hour angle.

In the triangle ZPM (Fig. 48),

given $PZ = 90° - l = \text{co-latitude},$
$ZM = 90° - h = \text{co-altitude},$
$\angle PZM = a = \text{azimuth};$

to find $PM = 90° - d = \text{polar distance},$
$\angle ZPM = t = \text{hour angle}.$

Draw $MQ \perp NS$, and put $ZQ = m$,
then, if $a < 90°,\ PQ = 90° - (l + m),$
and if $a > 90°,\ PQ = 90° - (l - m);$

and, by Napier's Rules,

$$\begin{aligned} \cos a &= \pm \tan m \tan h, \\ \sin d &= \cos PQ \cos MQ, \\ \sin h &= \cos m \cos MQ; \end{aligned}$$

whence,

$$\begin{aligned} \tan m &= \pm \cot h \cos a, \\ \sin d &= \sin h \sin(l \pm m) \sec m, \end{aligned}$$

in which the $-$ sign is to be used if $a > 90°$. The hour angle may then be found by means of [43], whence we have

$$\sin t = \sin a \cos h \sec d.$$

§ 67. PROBLEM.

To find the hour angle of a heavenly body when its declination, its altitude, and the latitude of the place are known.

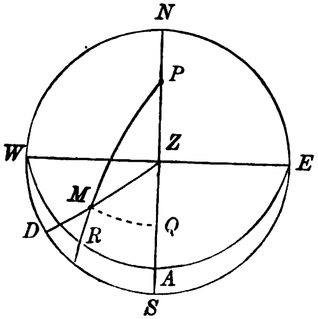


Fig. 49.

In the triangle ZPM (Fig. 49),

given $PZ = 90° - l,$
$PM = 90° - d = p,$
$ZM = 90° - h;$

required

$\angle ZPM = t.$

If, in the first formula of [46],

$$\sin \tfrac{1}{2} A = \sqrt{\sin(s-b)\sin(s-c)\csc b \csc c},$$

we put

$$A = t, \quad a = 90° - h, \quad b = p, \quad c = 90° - l,$$

we have

$$s - b = 90° - \tfrac{1}{2}(l + p + h), \quad s - c = \tfrac{1}{2}(l + p - h),$$

and the formula becomes

$$\sin \tfrac{1}{2} t = \pm \left[\cos \tfrac{1}{2}(l + p + h) \sin \tfrac{1}{2}(l + p - h) \sec l \csc p\right]^{\frac{1}{2}};$$

in which the − sign is to be taken when the body is east of the meridian.

If the body is the sun, how can the local time be found when the hour angle has been computed? (See § 65.)

§ 68. PROBLEM.

To find the altitude and azimuth of a celestial body, when its declination, its hour angle, and the latitude of the place are known.

In the triangle ZPM (Fig. 49),

given $PZ = 90^\circ - l,$

$PM = 90^\circ - d = p,$

$\angle ZPM = t;$

required $ZM = 90^\circ - h,$

$\angle PZM = a.$

Here there are given two sides and the included angle. Placing $PQ = m$, and proceeding as in § 66, we obtain

$$\begin{aligned}\tan m &= \cot d \cos t,\\ \sin h &= \sin (l + m) \sin d \sec m,\\ \tan a &= \sec (l + m) \tan t \sin m,\end{aligned}$$

in the last of which formulas a must be marked E. or W., to agree with the hour angle.

§ 69. PROBLEM.

To find the latitude of the place when the altitude of a celestial body, its declination, and its hour angle are known.

In the triangle ZPM (Fig. 49),

given $ZM = 90^\circ - h,$

$PM = 90^\circ - d,$

$\angle ZPM = t;$

required $PZ = 90^\circ - l.$

Let $PQ = m, \ ZQ = n.$

Then, by Napier's Rules,

$$\cos t = \tan m \tan d,$$
$$\sin h = \cos n \cos MQ,$$
$$\sin d = \cos m \cos MQ;$$

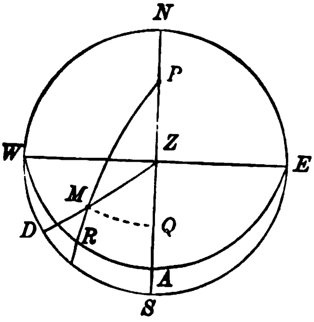


Fig. 49.

whence,

$$\tan m = \cot d \cos t,$$
$$\cos n = \cos m \sin h \csc d,$$

and it is evident from the figure that

$$l = 90^\circ - (m \pm n),$$

in which the sign + or the sign − is to be taken according as the body and the elevated pole are *on the same side* of the prime vertical or *on opposite sides.*

In fact, both values of l may be possible for the same altitude and hour angle; but, unless n is very small, the two values will differ largely from each other, so that the observer has no difficulty in deciding which of them should be taken.

§ 70. PROBLEM.

Given the declination, the right ascension of a star, and the obliquity of the ecliptic, to find the latitude and the longitude of the star.

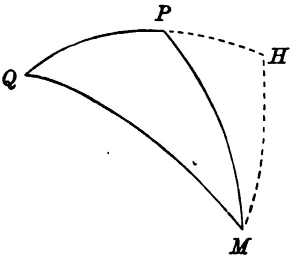


Fig. 50.

Let M (Fig. 50) be the star, P be the pole of the equinoctial, and Q the pole of the ecliptic.

Then, on the triangle PMQ,

given $PQ = e = 23^\circ 27'$,

$$PM = 90^\circ - d,$$
$$\angle MPQ = 90^\circ + r \text{ (see Fig. 47)};$$

required $QM = 90^\circ - u$, and $\angle PQM = 90^\circ - v$ (see Fig. 47).

In this case, also, two sides and the included angle are given. Draw $MH \perp PQ$, and meeting it produced at H, and let $PH = n$.

By Napier's Rules,

$$\sin r = \tan n \tan d,$$
$$\sin u = \cos(e+n) \cos MH,$$
$$\sin d = \cos n \cos MH,$$
$$\sin(e+n) = \tan v \tan MH,$$
$$\sin n = \tan r \tan MH;$$

whence,
$$\tan n = \cot d \sin r,$$
$$\sin u = \sin d \cos(e+n) \sec n,$$
$$\tan v = \tan r \sin(e+n) \csc n.$$

To avoid obtaining u from its sine we may proceed as follows:

From the last two equations we have, by division,

$$\sin u = \tan v \cot(e+n) \sin d \cot r \tan n.$$

By taking MH as middle part, successively, in the triangles MQH and MPH, we obtain

$$\cos u \cos v = \cos d \cos r;$$

whence,
$$\cos u = \sec v \cos d \cos r.$$

From these values of $\sin u$ and $\cos u$ we obtain, by division,

$$\tan u = \sin v \cot(e+n) \tan d \csc r \tan n.$$

From the relation

$$\sin r = \tan n \tan d,$$

it follows that $\tan d \csc r \tan n = 1$.

Therefore $\tan u = \sin v \cot(e+n)$,

a formula by which u can be easily found after v has been computed.

EXERCISE XXXII.

1. Find the dihedral angle made by the lateral faces of a regular ten-sided pyramid; given the angle $A = 18°$, made at the vertex by two adjacent lateral edges.

2. Through the foot of a rod which makes the angle A with a plane, a straight line is drawn in the plane. This line makes the angle B with the projection of the rod upon the plane. What angle does this line make with the rod?

3. Find the volume V of an oblique parallelopipedon; given the three unequal edges a, b, c, and the three angles l, m, n, which the edges make with one another.

4. The continent of Asia has nearly the shape of an equilateral triangle, the vertices being the East Cape, Cape Romania, and the Promontory of Baba. Assuming each side of this triangle to be 4800 geographical miles, and the earth's radius to be 3440 geographical miles, find the area of the triangle: (i.) regarded as a plane triangle; (ii.) regarded as a spherical triangle.

5. A ship sails from a harbor in latitude l, and keeps on the arc of a great circle. Her *course* (or angle between the direction in which she sails and the meridian) at starting is a. Find where she will cross the equator, her course at the equator, and the distance she has sailed.

6. Two places have the same latitude l, and their distance apart, measured on an arc of a great circle, is d. How much greater is the arc of the parallel of latitude between the places than the arc of the great circle? Compute the results for $l = 45°$, $d = 90°$.

7. The shortest distance d between two places and their latitudes l and l' are known. Find the difference between their longitudes.

8. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth; show how to find the area of the spherical triangle formed by arcs of great circles passing through the places.

9. The distance between Paris and Berlin (that is, the arc of a great circle between these places) is equal to 472 geographical miles. The latitude of Paris is 48° 30′ 13″; that of Berlin, 52° 30′ 16″. When it is noon at Paris what time is it at Berlin?

NOTE. Owing to the apparent motion of the sun, the local time over the earth's surface at any instant varies at the rate of one hour for 15° of longitude; and the more *easterly* the place, the *later* the local time.

10. The altitude of the pole being 45°, I see a star on the horizon and observe its azimuth to be 45°; find its polar distance.

11. Given the latitude l of the observer, and the declination d of the sun; find the local time (apparent solar time) of sunrise and sunset, and also the azimuth of the sun at these times (refraction being neglected). When and where does the sun rise on the longest day of the year (at which time $d = +23° 27'$) in Boston ($l = 42° 21'$), and what is the length of the day from sunrise to sunset? Also, find when and where the sun rises in Boston on the shortest day of the year (when $d = -23° 27'$), and the length of this day.

12. When is the solution of the problem in Example 11 impossible, and for what places is the solution impossible?

13. Given the latitude of a place and the sun's declination; find his altitude and azimuth at 6 o'clock A.M. (neglecting refraction). Compute the results for the longest day of the year at Munich ($l = 48° 9'$).

14. How does the altitude of the sun at 6 A.M. *on a given day* change as we go from the equator to the pole? At what time of the year is it a maximum *at a given place?* (Given $\sin h = \sin l \sin d$.)

15. Given the latitude of a place north of the equator, and the declination of the sun; find the time of day when the sun bears due east and due west. Compute the results for the longest day at St. Petersburg ($l = 59° 56'$).

16. Apply the general result in Example 15 ($\cos t = \cot l \tan d$) to the case when the days and nights are equal in length (that is, when $d = 0°$). Why can the sun in summer never be due east before 6 A.M., or due west after 6 P.M.? How does the time of bearing due east and due west change with the declination of the sun? Apply the general result to the cases where $l < d$ and $l = d$. What does it become at the north pole?

17. Given the sun's declination and his altitude when he bears due east; find the latitude of the observer.

18. At a point O in a horizontal plane MN a staff OA is fixed, so that its angle of inclination AOB with the plane is equal to the latitude of the place, 51° 30′ N., and the direction OB is due north. What angle will OB make with the shadow of OA on the plane, at 1 P.M., when the sun is on the equinoctial?

19. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

20. At a certain place the sun is observed to rise exactly in the north-east point on the longest day of the year; find the latitude of the place.

21. Find the latitude of the place at which the sun sets at 10 o'clock on the longest day.

22. To what does the general formula for the hour angle, in § 67, reduce when (i.) $h = 0°$, (ii.) $l = 0°$ and $d = 0°$, (iii.) l or $d = 90°$?

23. What does the general formula for the azimuth of a celestial body, in § 68, become when $t = 90° = 6$ hours?

24. Show that the formulas of § 69, if $t = 90°$, lead to the equation $\sin l = \sin h \csc d$; and that if $d = 0°$, they lead to the equation $\cos l = \sin h \sec t$.

25. Given latitude of place 52° 30′ 16″, declination of star 38°, its hour angle 28° 17′ 15″; find its altitude.

26. Given latitude of place 51° 19′ 20″, polar distance of star 67° 59′ 5′, its hour angle 15° 8′ 12″; find its altitude and its azimuth.

27. Given the declination of a star 7° 54′, its altitude 22° 45′ 12″, its azimuth 129° 45′ 37″; find its hour angle and the latitude of the observer.

28. Given the longitude v of the sun, and the obliquity of the ecliptic $e = 23° 27'$; find the declination d, and the right ascension r.

29. Given the obliquity of the ecliptic $e = 23° 27'$, the latitude of a star 51°, its longitude 315°; find its declination and its right ascension.

30. Given the latitude of place 44° 50′ 14″, the azimuth of a star 138° 58′ 43″, and its hour angle 20°; find its declination.

31. Given latitude of place 51° 31′ 48″, altitude of sun west of the meridian 35° 14′ 27″, its declination +21° 27′; find the local apparent time.

32. Given latitude of place l, the polar distance p of a star, and its altitude h; find its azimuth a.

APPENDIX.

FORMULAS.

$$\left.\begin{array}{ll} 1. & \sin^2 A + \cos^2 A = 1. \\ 2. & \tan A = \dfrac{\sin A}{\cos A}. \\ 3. & \left\{\begin{array}{l} \sin A \times \csc A = 1. \\ \cos A \times \sec A = 1. \\ \tan A \times \cot A = 1. \end{array}\right. \end{array}\right\} \S\ 5.$$

$$\left.\begin{array}{ll} 4. & \sin(x+y) = \sin x \cos y + \cos x \sin y. \\ 5. & \cos(x+y) = \cos x \cos y - \sin x \sin y. \\ 6. & \tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}. \\ 7. & \cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}. \end{array}\right\} \S\ 31.$$

$$\left.\begin{array}{ll} 8. & \sin(x-y) = \sin x \cos y - \cos x \sin y. \\ 9. & \cos(x-y) = \cos x \cos y + \sin x \sin y. \\ 10. & \tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}. \\ 11. & \cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}. \end{array}\right\} \S\ 32.$$

$$\left.\begin{array}{ll} 12. & \sin 2x = 2 \sin x \cos x. \\ 13. & \cos 2x = \cos^2 x - \sin^2 x. \end{array}\right\} \S\ 33.$$

$$\left.\begin{array}{ll} \textbf{14.} & \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}. \\[2ex] \textbf{15.} & \cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}. \end{array}\right\}\ \S\ 33.$$

$$\left.\begin{array}{ll} \textbf{16.} & \sin \tfrac{1}{2} z = \pm\sqrt{\dfrac{1-\cos z}{2}}. \\[2ex] \textbf{17.} & \cos \tfrac{1}{2} z = \pm\sqrt{\dfrac{1+\cos z}{2}}. \\[2ex] \textbf{18.} & \tan \tfrac{1}{2} z = \pm\sqrt{\dfrac{1-\cos z}{1+\cos z}}. \\[2ex] \textbf{19.} & \cot \tfrac{1}{2} z = \pm\sqrt{\dfrac{1+\cos z}{1-\cos z}}. \end{array}\right\}\ \S\ 34.$$

$$\left.\begin{array}{ll} \textbf{20.} & \sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B). \\[2ex] \textbf{21.} & \sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B). \\[2ex] \textbf{22.} & \cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B). \\[2ex] \textbf{23.} & \cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B). \\[2ex] \textbf{24.} & \dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \end{array}\right\}\ \S\ 35.$$

25. $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}.$ § 36.

26. $a^2 = b^2 + c^2 - 2bc \cos A.$ § 37.

27. $\dfrac{a-b}{a+b} = \dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$ § 38.

28. $\sin \tfrac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$ § 43.

$$\left.\begin{array}{ll}
\textbf{29.} & \cos\frac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}. \\[2ex]
\textbf{30.} & \tan\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \\[2ex]
\textbf{31.} & \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=r. \\[2ex]
\textbf{32.} & \tan\frac{1}{2}A=\frac{r}{s-a}.
\end{array}\right\}\ \S\ 43.$$

$$\left.\begin{array}{ll}
\textbf{33.} & F=\frac{1}{2}ac\sin B. \\[2ex]
\textbf{34.} & F=\sqrt{s(s-a)(s-b)(s-c)}. \\[2ex]
\textbf{35.} & F=\frac{abc}{4R}. \\[2ex]
\textbf{36.} & F=\frac{1}{2}r(a+b+c)=rs.
\end{array}\right\}\ \S\ 44.$$

SPHERICAL TRIGONOMETRY.

$$\left.\begin{array}{ll}
\textbf{37.} & \cos c=\cos a\cos b. \\[2ex]
\textbf{38.} & \begin{cases}\sin a=\sin c\sin A.\\ \sin b=\sin c\sin B.\end{cases} \\[2ex]
\textbf{39.} & \begin{cases}\cos A=\tan b\cot c.\\ \cos B=\tan a\cot c.\end{cases} \\[2ex]
\textbf{40.} & \begin{cases}\cos A=\cos a\sin B.\\ \cos B=\cos b\sin A.\end{cases} \\[2ex]
\textbf{41.} & \begin{cases}\sin b=\tan a\cot A.\\ \sin a=\tan b\cot B.\end{cases} \\[2ex]
\textbf{42.} & \cos c=\cot A\cot B.
\end{array}\right\}\ \S\ 46.$$

$$\textbf{43.}\ \left\{\begin{array}{l}
\sin a\sin B=\sin b\sin A.\\
\sin a\sin C=\sin c\sin A.\\
\sin b\sin C=\sin c\sin B.
\end{array}\right\}\ \S\ 51.$$

44.
$$\left\{\begin{array}{l}\cos a = \cos b \cos c + \sin b \sin c \cos A.\\ \cos b = \cos a \cos c + \sin a \sin c \cos B.\\ \cos c = \cos a \cos b + \sin a \sin b \cos C.\end{array}\right.$$

45.
$$\left\{\begin{array}{l}\cos A = -\cos B \cos C + \sin B \sin C \cos a.\\ \cos B = -\cos A \cos C + \sin A \sin C \cos b.\\ \cos C = -\cos A \cos B + \sin A \sin B \cos c.\end{array}\right.$$

§ 51.

46.
$$\left\{\begin{array}{l}\sin \tfrac{1}{2} A = \sqrt{\sin (s-b) \sin (s-c) \csc b \csc c}.\\ \cos \tfrac{1}{2} A = \sqrt{\sin s \sin (s-a) \csc b \csc c}.\\ \tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s-a) \sin (s-b) \sin (s-c)}.\end{array}\right.$$

47.
$$\left\{\begin{array}{l}\sin \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \csc B \csc C}.\\ \cos \tfrac{1}{2} a = \sqrt{\cos (S-B) \cos (S-C) \csc B \csc C}.\\ \tan \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \sec (S-B) \sec (S-C)}.\end{array}\right.$$

§ 52.

48.
$$\left\{\begin{array}{l}\cos \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} C.\\ \sin \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2} (a-b) \cos \tfrac{1}{2} C.\\ \cos \tfrac{1}{2} (A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} C.\\ \sin \tfrac{1}{2} (A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a-b) \cos \tfrac{1}{2} C.\end{array}\right.$$

49.
$$\left\{\begin{array}{l}\tan \tfrac{1}{2} (A+B) = \dfrac{\cos \tfrac{1}{2} (a-b)}{\cos \tfrac{1}{2} (a+b)} \cot \tfrac{1}{2} C.\\[2ex] \tan \tfrac{1}{2} (A-B) = \dfrac{\sin \tfrac{1}{2} (a-b)}{\sin \tfrac{1}{2} (a+b)} \cot \tfrac{1}{2} C.\\[2ex] \tan \tfrac{1}{2} (a+b) = \dfrac{\cos \tfrac{1}{2} (A-B)}{\cos \tfrac{1}{2} (A+B)} \tan \tfrac{1}{2} c.\\[2ex] \tan \tfrac{1}{2} (a-b) = \dfrac{\sin \tfrac{1}{2} (A-B)}{\sin \tfrac{1}{2} (A+B)} \tan \tfrac{1}{2} c.\end{array}\right.$$

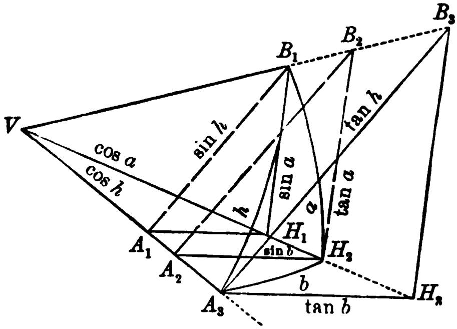
§ 53.

50. $$F = \frac{E}{180^\circ} \pi R^2.$$

51. $$\tan^2 \tfrac{1}{4} E = \tan \tfrac{1}{2} (s-a) \tan \tfrac{1}{2} (s-b) \tan \tfrac{1}{2} (s-c).$$

§ 60.

Prof. Blakslee's construction by which the direction ratios for plane right triangles give directly from a figure the analogies for a right trihedral or for a right spherical triangle.



The construction consists of two parts.

(*a*) Lay off from the vertex V a unit's distance on each edge.

(*b*) Pass through the three extremities of these distances three planes perpendicular to one of the edges, as VA. Now these three parallel planes will cut out three similar right triangles. The first being constructed in either of the two usual ways, the construction of the others is evident.

Since the plane angles A_1, A_2, A_3 all equal the dihedral A, and the nine right triangles in the three faces give the values in the figure, we have:

(1) $\sin A = \sin a : \sin h$; similarly, $\sin B = \sin b : \sin h$.

(2) $\cos A = \tan b : \tan h$; similarly, $\cos B = \tan a : \tan h$.

(3) $\tan A = \tan a : \sin b$; similarly, $\tan B = \tan b : \sin a$.

(4) $\cos h = \cos a \ \cos b$; (by 3) $= \cot A \ \cot B$.

(5) $\sin A = \cos B : \cos b$; $\sin B = \cos A : \cos a$.

Note. If a sphere of unit radius be described about V as a centre, the three faces will cut out a right spherical triangle, having the sides a, b, and h, and angles A, B, and H. The above formulas are thus seen to be the analogies of:

(1) $\sin A = a : h$; $\sin B = b : h$.

(2) $\cos A = b : h$; $\cos B = a : h$.

(3) $\tan A = a : b$; $\tan B = b : a$.

(4) $h^2 = a^2 + b^2$; $1 = \sin^2 + \cos^2$; $1 = \cot A \cot B$.

(5) $\sin A = \cos B$; $\sin B = \cos A$.

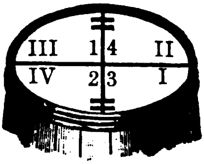
Napier's rules give only the following, which follow from the analogies as numbered:

$$\begin{array}{l}\text{By}\\ (1)\end{array}\left\{\begin{array}{l}\sin a = \sin A \sin h = \tan b \cot B\\ \sin b = \sin B \sin h = \tan a \cot A\end{array}\right\}(3)$$

$$(5)\left\{\begin{array}{l}\cos A = \sin B \cos a = \tan b \cot h\\ \cos B = \sin A \cos b = \tan a \cot h\end{array}\right\}(2)$$

$$(4)\ \{\cos h = \cos a \cos b = \cot A \cot B\}\ (4)$$

THE GAUSS EQUATIONS.



$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C.$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C.$$

$$f \tfrac{1}{2}(A \pm B) \quad f \tfrac{1}{2} c = \quad f \tfrac{1}{2}(a \pm b) \quad f \tfrac{1}{2} C.$$

RULE I. sin in (I.) gives − in (3), and conversely.
cos in (I.) gives + in (3), and conversely.

RULE II. Functions have same names in (2) and (3).
Functions have co-names in (4) and (1).

ANSWERS.

PLANE TRIGONOMETRY.

EXERCISE I.

1. $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$, $\tan B = \frac{b}{a}$, $\cot B = \frac{a}{b}$, $\sec B = \frac{c}{a}$, $\csc B = \frac{c}{b}$

3. (i.) $\sin = \frac{3}{5}$, $\cos = \frac{4}{5}$, (ii.) $\sin = \frac{5}{13}$, etc. (v.) $\sin = \frac{28}{53}$, etc.
$\tan = \frac{3}{4}$, $\cot = \frac{4}{3}$, (iii.) $\sin = \frac{8}{17}$, etc. (vi.) $\sin = \frac{119}{169}$, etc.
$\sec = \frac{5}{4}$, $\csc = \frac{5}{3}$. (iv.) $\sin = \frac{9}{41}$, etc.

4. The required condition is that $a^2 + b^2 = c^2$. It is.

5. (i.) $\sin = \frac{2mn}{m^2+n^2}$, etc. (iii.) $\sin = \frac{q}{s}$, etc.
(ii.) $\sin = \frac{2xy}{x^2+y^2}$, etc. (iv.) $\sin = \frac{ms}{qr}$, etc.

7. In (iii.) $p^2q^2 + q^2s^2 = p^2s^2$; in (iv.) $m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2$.

8. $c = 145$; whence, $\sin A = \frac{24}{145} = \cos B$; $\cos A = \frac{143}{145} = \sin B$;
$\tan A = \frac{24}{143} = \cot B$; $\cot A = \frac{143}{24} = \tan B$; $\sec A = \frac{145}{143} = \csc B$; etc.

9. $b = 0.023$; whence, $\tan A = \cot B = \frac{23}{264}$; $\cot A = \tan B = \frac{264}{23}$, etc.

10. $a = 16.8$; whence, $\sin A = \frac{168}{193} = \cos B$, etc.

11. $c = p + q$; whence, $\sin A = \frac{\sqrt{p^2+q^2}}{p+q} = \cos B$; etc.

12. $b = \sqrt{q(p+q)}$; whence, $\tan A = \sqrt{\frac{p}{q}} = \cot B$; etc.

13. $a = p - q$; whence, $\sin A = \frac{p-q}{p+q} = \cos B$; etc.

14. $\sin A = \frac{2}{5}\sqrt{5} = 0.89443$; etc. 15. $\sin A = \frac{2}{3}$; etc.

16. $\sin A = \frac{1}{8}(5 + \sqrt{7}) = 0.95572$; etc.

17. $\cos A = \frac{1}{8}(\sqrt{31} - 1) = 0.57097$; $\sin A = \frac{1}{8}(\sqrt{31} + 1) = 0.82097$; etc.

18. $a = 12.3$. 20. $a = 9$. 22. $c = 40$.

19. $b = 1.54$. 21. $b = 68$. 23. $c = 229.62$.

24. Construct a rt. △ with legs equal to 3 and 2 respectively; then construct a similar △ with hypotenuse equal to 6.
In like manner, 25, 26, 27, may be solved.

28. $a = 1.5$ miles; $b = 2$ miles. 31. 400,000 miles.

30. $a = 0.342$, $b = 0.940$; $a = 1.368$, $b = 3.760$. 32. 142.926 yards.

EXERCISE II.

5. Through A (Fig. 3) draw a tangent, and take $AT = 3$; the angle AOT is the required angle.
6. From O (Fig. 3) as a centre, with a radius = 2, describe an arc cutting at S the tangent drawn through B; the angle SOA is the required angle.
7. In Fig. 3, take $OM = \frac{1}{2}$, and erect $MP \perp OA$ and intersecting the circumference at P; the angle POM is the required angle.
8. Since $\sin x = \cos x$, $OM = PM$ (Fig. 3), and $x = 45°$; hence, construct $x = 45°$.
9. Construct a rt. △ with one leg = twice the other; the angle opposite the longer leg is the required angle.
10. Divide OA (Fig. 3) into four equal parts; at the first point of division from O erect a perpendicular to meet the circumference at some point P. Join OP; the angle AOP is the required angle.

21. $r \sin x$. 22. Leg adjacent to $A = nc$, leg opposite to $A = mc$.

EXERCISE III.

1. $\cos 60°$. $\cot 1°$. $\sec 71° 50'$. $\tan 7° 41'$.
   $\sin 45°$. $\tan 75°$. $\sin 52° 36'$. $\sec 35° 14'$.
2. $\cos 30°$. $\cot 33°$. $\sec 20° 58'$. $\tan 0° 1'$.
   $\sin 15°$. $\tan 6°$. $\sin 4° 21'$. $\sec 44° 59$.
3. $\frac{1}{3}\sqrt{3}$.
4. $\tan A = \cot A = \cot(90° - A)$; hence, $A = 90° - A$ and $A = 45°$.

5. 30°. 7. 90°. 9. 22° 30′. 11. 10°.

6. 30° 8. 60°. 10. 18°. 12. $\frac{90°}{n+1}$.

EXERCISE V.

1. $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\cot A = \frac{5}{12}$, $\sec A = \frac{13}{5}$, $\csc A = \frac{13}{12}$.
2. $\cos A = 0.6$, $\tan A = 1.3333$, $\cot A = 0.75$, $\sec A = 1.6667$, $\csc A = 1.25$.
3. $\sin A = \frac{11}{61}$, $\tan A = \frac{11}{60}$, $\cot A = \frac{60}{11}$, $\sec A = \frac{61}{60}$, $\csc A = \frac{61}{11}$.
4. $\sin A = 0.96$, $\tan A = 3.42854$, $\cot A = 0.29167$, $\sec A = 3.5714$.
5. $\sin A = 0.8$, $\cos A = 0.6$, $\cot A = 0.75$, $\sec A = 1.6667$, $\csc A = 1.25$.
6. $\sin A = \frac{1}{2}\sqrt{2}$, $\cos A = \frac{1}{2}\sqrt{2}$, $\tan A = 1$, $\sec A = \sqrt{2}$, $\csc A = \sqrt{2}$.
7. $\tan A = 2$, $\sin A = 0.90$, $\cos A = 0.45$, $\sec A = 2.22$, $\csc A = 1.11$.

8. $\cos A = \frac{1}{2}$, $\sin A = \frac{1}{2}\sqrt{3}$, $\tan A = \sqrt{3}$, $\cot A = \frac{1}{3}\sqrt{3}$, $\csc A = \frac{2}{3}\sqrt{3}$.

9. $\sin A = \frac{1}{2}\sqrt{2}$, $\cos A = \frac{1}{2}\sqrt{2}$, $\tan A = 1$, $\cot A = 1$, $\sec A = \sqrt{2}$.

10. $\cos A = \sqrt{1-m^2}$, $\tan A = \frac{m}{1-m^2}\sqrt{1-m^2}$, $\cot A = \frac{1}{m}\sqrt{1-m^2}$.

11. $\cos A = \frac{1-m^2}{1+m^2}$, $\tan A = \frac{2m}{1-m^2}$, $\cot A = \frac{1-m^2}{2m}$, $\sec A = \frac{1+m^2}{1-m^2}$.

12. $\sin A = \frac{m^2-n^2}{m^2+n^2}$, $\tan A = \frac{m^2-n^2}{2mn}$, $\sec A = \frac{m^2+n^2}{2mn}$.

13. $\cot = 1$, $\sin = \frac{1}{2}\sqrt{2}$, $\cos = \frac{1}{2}\sqrt{2}$, $\sec = \sqrt{2}$, $\csc = \sqrt{2}$.

14. $\cos = \frac{1}{2}\sqrt{3}$, $\tan = \frac{1}{3}\sqrt{3}$, $\cot = \sqrt{3}$, $\sec = \frac{2}{3}\sqrt{3}$, $\csc = 2$.

15. $\sin = \frac{1}{2}\sqrt{3}$, $\cos = \frac{1}{2}$, $\tan = \sqrt{3}$, $\cot = \frac{1}{3}\sqrt{3}$, $\sec = 2$.

16. $\sin = \frac{1}{2}\sqrt{2-\sqrt{3}}$, $\cos = \frac{1}{2}\sqrt{2+\sqrt{3}}$, $\cot = 2+\sqrt{3}$.

17. $\sin = \frac{1}{2}\sqrt{2-\sqrt{2}}$, $\cos = \frac{1}{2}\sqrt{2+\sqrt{2}}$, $\tan = \sqrt{2}-1$.

18. $\cos = 1$, $\tan = 0$, $\cot = \infty$, $\sec = 1$, $\csc = \infty$.

19. $\cos = 0$, $\tan = \infty$, $\cot = 0$, $\sec = \infty$, $\csc = 1$.

20. $\sin = 1$, $\cos = 0$, $\cot = 0$, $\sec = \infty$, $\csc = 1$.

21. $\cos A = \sqrt{1-\sin^2 A}$, $\tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}}$, $\csc A = \frac{1}{\sin A}$.

22. $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A = \frac{\cos A}{\sqrt{1-\cos^2 A}}$,
$\sec A = \frac{1}{\cos A}$, $\csc A = \frac{1}{\sqrt{1-\cos^2 A}}$.

23. $\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A = \frac{1}{\sqrt{1+\tan^2 A}}$, $\cot A = \frac{1}{\tan A}$,
$\sec A = \sqrt{1+\tan^2 A}$, $\csc A = \frac{\sqrt{1+\tan^2 A}}{\tan A}$.

24. $\tan A = \frac{1}{\cot A}$, $\csc A = \sqrt{1+\cot^2 A}$, $\sin A = \frac{1}{\sqrt{1+\cot^2 A}}$,
$\cos A = \frac{\cot A}{\sqrt{1+\cot^2 A}}$, $\sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}$.

25. $\sin A = \frac{1}{5}\sqrt{5}$, $\cos A = \frac{2}{5}\sqrt{5}$.

26. $\sin A = \frac{1}{4}\sqrt{15}$, $\tan A = \sqrt{15}$.

27. $\sin A = \frac{9}{41}$, $\cos A = \frac{40}{41}$.

28. $\frac{1-3\cos^2 A+3\cos^4 A}{\cos^2 A-\cos^4 A}$.

EXERCISE VI.

1. $\frac{b}{c} = \cos A$; $\therefore c = \frac{b}{\cos A}$.

5. $A = 90° - B$, $a = c \cos B$, $b = c \sin B$.

39. $c = 7.8112$, $A = 39° 48'$, $B = 50° 12'$.

40. $b = 69.99$, $A = 30' 12''$, $B = 89° 29' 48''$.

41. $a = 1.1886$, $A = 43° 20'$, $B = 46° 40'$.

42. $b = 21.249$, $c = 22.372$, $B = 71° 46'$.

43. $a = 6.6882$, $c = 13.738$, $B = 60° 52'$.

44. $a = 63.89$, $b = 23.369$, $B = 20° \ 6'$.

45. $a = 19.40$, $b = 18.778$, $A = 45° 56'$.

46. $b = 53.719$, $c = 71.377$, $A = 41° 11'$.

47. $a = 12.981$, $c = 15.796$, $A = 55° 16'$.

48. $a = 0.58046$, $b = 8.442$, $A = 3° 56'$.

49. $F = \frac{1}{2}(c^2 \sin A \cos A)$.

50. $F = \frac{1}{2}(a^2 \cot A)$.

51. $F = \frac{1}{2}(b^2 \tan A)$.

52. $F = \frac{1}{2}(a\sqrt{c^2 - a^2})$.

53. $b = 11.6$, $c = 15.315$, $A = 40° 45' 48''$, $B = 49° 14' 12''$.

54. $a = 7.2$, $c = 8.766$, $B = 34° 46' 40''$, $A = 55° 13' 20''$.

55. $a = 3.6474$, $b = 6.58$, $c = 7.5233$, $B = 61°$.

56. $a = 10.283$, $b = 19.449$, $A = 27° 52'$, $B = 62° 8'$.

57. 19° 28′ and 70° 32′.

58. 3 and 5.1961.

59. $a = c \cos \frac{90°}{n+1}$, $b = c \sin \frac{90°}{n+1}$.

60. 36° 52′ 12″ and 53° 7′ 48″.

61. 212.1 feet.

62. 732.22 feet.

63. 3270 feet.

64. 37.3 feet, 96 feet.

65. $\tan A = \frac{a}{b}$, $A = 59° 45'$.

66. $a = b \tan A$, 95.34.

67. 1° 25′ 56″.

68. 7.0712 miles in each direction.

69. 20.88 feet.

70. 56.65 feet.

71. 228.63 yards.

72. 136.6 feet.

73. 140 feet.

74. 84.74 feet.

EXERCISE VII.

1. $C = 2(90° - A)$, $c = 2a \cos A$, $h = a \sin A$.

2. $A = \frac{1}{2}(180° - C)$, $c = 2a \cos A$, $h = a \sin A$.

3. $C = 2(90° - A)$, $a = c \div 2 \cos A$, $h = a \sin A$.

4. $A = \frac{1}{2}(180^\circ - C)$, $a = c \div 2\cos A$, $h = a\sin A$.
5. $C = 2\ (90^\circ - A)$, $a = h \div \sin A$, $c = 2a\cos A$.
6. $A = \frac{1}{2}(180^\circ - C)$, $a = h \div \sin A$, $c = 2a\cos A$.
7. $\sin A = h \div a$, $C = 2(90^\circ - A)$, $c = 2a\cos A$.
8. $\tan A = h \div \frac{1}{2}c$, $C = 2(90^\circ - A)$, $a = h \div \sin A$.
9. $A = 67^\circ\, 22'\, 50''$, $C = 45^\circ\, 14'\, 20''$, $h = 13.2$.
10. $c = 0.21943$, $h = 0.27384$, $F = 0.03004$.
11. $a = 2.055$, $h = 1.6852$, $F = 1.9819$.
12. $a = 7.705$, $c = 3.6676$, $F = 13.73$.
13. $A = 79^\circ\, 36'\, 30''$, $C = 20^\circ\, 45'\, 30''$, $c = 2.4206$.
14. $A = 77^\circ\, 19'\, 11''$, $C = 25^\circ\, 21'\, 38''$, $a = 20.5$.
15. $A = 25^\circ\, 28'$, $C = 129^\circ\, 4'$, $a = 81.388$, $h = 35$.
16. $A = 81^\circ\, 12'\, 9''$, $C = 17^\circ\, 35'\, 42''$, $a = 17$, $c = 5.2$.

17. $F = \frac{1}{4}c\sqrt{4a^2 - c^2}$.
18. $F = a^2 \sin\frac{1}{2}C \cos\frac{1}{2}C$.
19. $F = a^2 \sin A \cos A$.
20. $F = h^2 \tan\frac{1}{2}C$.
21. 28.284 feet, 4525.44 sq. feet.
22. 0.76537.
23. 94° 20′.
24. 2.7261.
25. 38° 56′ 33″.
26. 37.7

EXERCISE VIII.

1. $r = 1.618$, $h = 1.5388$, $F = 7.694$.
2. $r = 11.269$, $h = 10.886$, $F = 381.04$.
3. $h = 0.9848$, $p = 6.2514$, $F = 3.0782$.
4. $h = 19.754$, $c = 6.2537$, $F = 1236$.
5. $r = 1.0824$, $c = 0.8284$, $F = 3.3137$.
6. $r = 2.592$, $h = 2.488$, $c = 1.4615$.
7. $r = 1.5994$, $h = 1.441$, $p = 9.716$.

8. 0.6181.
9. 0.64984.
10. 0.51764.
11. $b = \dfrac{c}{2\cos\dfrac{90^\circ}{n}}$.
12. 0.2238.
13. 0.31.
14. 0.82842.
15. 94.63.
16. 415.
17. 11.636.
18. 99.64.
19. 1.0235.
20. 0.635.

EXERCISE IX.

5. Two angles: one in Quadrant I., the other in Quadrant II.

6. Four values: two in Quadrant I., two in Quadrant IV.

7. x may have two values in the first case, and one value in each of the other cases.

8. If $\cos x = -\frac{2}{3}$, x is between 90° and 270°; if $\cot x = 4$, x is between 0° and 90° or 180° and 270°; if $\sec x = 80$, x is between 0° and 90° or between 270° and 360°; if $\csc x = -3$, x is between 180° and 360°.

9. In Quadrant III.; in Quadrant II.; in Quadrant III.

10. 40 angles; 20 positive and 20 negative.

11. +, when x is known to be in Quadrant I. or IV.; −, when x is known to be in Quadrant II. or III.

14. $\sin x = -\frac{4}{7}\sqrt{3}$, $\tan x = -4\sqrt{3}$, $\cot x = -\frac{1}{12}\sqrt{3}$, $\csc x = -\frac{7}{12}\sqrt{3}$.

15. $\sin x = \pm\frac{1}{10}\sqrt{10}$, $\cos x = \mp\frac{3}{10}\sqrt{10}$, $\tan x = -\frac{1}{3}$, $\sec x = \mp\frac{1}{3}\sqrt{10}$.
$\csc x = \pm\sqrt{10}$.

16. The cosine, the tangent, the cotangent, and the secant are negative when the angle is obtuse.

17. Sine and cosecant leave it doubtful whether the angle is an acute angle or an obtuse angle; the other functions, if + determine an acute angle, if − an obtuse angle.

20. $\sin 450 = \sin(360 + 90) = \sin 90 = 1$; $\tan 540° = \tan 180° = 0$;
$\cos 630° = \cos 270° = 0$; $\cot 720° = \cot 0° = \infty$;
$\sin 810° = \sin 90° = 1$; $\csc 900° = \csc 180° = \infty$.

21. 45°, 135°, 225°, 315°. 22. 0. 23. 0. 24. 0.

25. $a^2 - b^2 + 4ab$.

EXERCISE X.

2. $\sin 172° = \sin 8°$.
3. $\cos 100° = -\sin 10°$.
4. $\tan 125° = -\cot 35°$.
5. $\cot 91° = -\tan 1°$.
6. $\sec 110° = -\csc 20°$.
7. $\csc 157° = \csc 23°$.
8. $\sin 204° = -\sin 24°$.
9. $\cos 359° = \cos 1°$.
10. $\tan 300° = -\cot 30°$.
11. $\cot 264° = \tan 6°$.
12. $\sec 244° = -\csc 26°$.
13. $\csc 271° = -\sec 1°$.
14. $\sin 163° 49' = \sin 16° 11'$.
15. $\cos 195° 33' = -\cos 15° 33'$.
16. $\tan 269° 15' = \cot 0° 45'$.
17. $\cot 139° 17' = -\cot 40° 43'$.
18. $\sec 299° 45' = \csc 29° 45'$.
19. $\csc 92° 25' = \sec 2° 25'$.

20. $\sin(-75^\circ) = -\sin 75^\circ = -\cos 15^\circ$, $\cos(-75^\circ) = \cos 75^\circ = \sin 15^\circ$, etc.

21. $\sin(-127^\circ) = -\sin 127^\circ = -\cos 37^\circ$, $\cos(-127^\circ) = \cos 127^\circ = -\sin 37^\circ$, etc.

22. $\sin(-200^\circ) = \sin 160^\circ = \sin 20^\circ$, $\cos(-200^\circ) = \cos 200^\circ = -\cos 20^\circ$, etc.

23. $\sin(-345^\circ) = -\sin 345^\circ = \sin 15^\circ$, $\cos(-345^\circ) = \cos 345^\circ = \cos 15^\circ$. etc.

24. $\sin(-52^\circ 37') = -\sin 52^\circ 37' = -\cos 37^\circ 23'$,
$\cos(-52^\circ 37') = \cos 52^\circ 37' = \sin 37^\circ 23'$, etc.

25. $\sin(-196^\circ 54') = \sin 196^\circ 54' = \cos 16^\circ 54'$,
$\cos(-196^\circ 54') = \cos 196^\circ 54' = -\cos 16^\circ 54'$, etc.

26. $\sin 120^\circ = \frac{1}{2}\sqrt{3}$, $\cos 120^\circ = -\frac{1}{2}$, etc.

27. $\sin 135^\circ = +\frac{1}{2}\sqrt{2}$, $\cos 135^\circ = -\frac{1}{2}\sqrt{2}$, etc.

28. $\sin 150^\circ = +\frac{1}{2}$, $\cos 150^\circ = -\frac{1}{2}\sqrt{3}$, etc.

29. $\sin 210^\circ = -\frac{1}{2}$, $\cos 210^\circ = -\frac{1}{2}\sqrt{3}$, etc.

30. $\sin 225^\circ = -\frac{1}{2}\sqrt{2}$, $\cos 225^\circ = -\frac{1}{2}\sqrt{2}$, etc.

31. $\sin 240^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 240^\circ = -\frac{1}{2}$, etc.

32. $\sin 300^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 300^\circ = +\frac{1}{2}$, etc.

33. $\sin(-30^\circ) = -\frac{1}{2}$, $\cos(-30^\circ) = +\frac{1}{2}\sqrt{3}$, etc

34. $\sin(-225^\circ) = +\frac{1}{2}\sqrt{2}$, $\cos(-225^\circ) = -\frac{1}{2}\sqrt{2}$, etc.

35. $\cos x = -\frac{1}{2}\sqrt{2}$ or $-\sqrt{\frac{1}{2}}$, etc., $x = 225^\circ$.

36. $\tan x = -\sqrt{\frac{1}{3}}$, $\sin x = \frac{1}{2}$, $\cos x = -\frac{1}{2}\sqrt{3}$, $x = 150^\circ$.

37. $\sin 3540^\circ = \sin 300^\circ = -\sin 60^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 3540^\circ = +\frac{1}{2}$, etc.

38. 210° and 330°; 120° and 300°.

39. 135°, 225°, and -225°; 150° and -30°.

40. 30°, 150°, 390°, and 510°.

41. $\sin 168^\circ$, $\cos 334^\circ$, $\tan 225^\circ$, $\cot 252^\circ$,
$\sin 349^\circ$, $\cos 240^\circ$, $\tan 64^\circ$, $\cot 177^\circ$.

42. 0.848. (Hint: $\tan 238^\circ = \tan 58^\circ$, $\sin 122^\circ = \sin 58^\circ$).

43. -1.952.

44. $(a - b)\sin x$.

45. $m \sin x \cos x$.

46. $(a - b)\cot x - (a + b)\tan x$.

47. $a^2 + b^2 + 2ab\cos x$.

48. 0.

49. $\cos x \sin y - \sin x \cos y$.

50. $\tan x$.

51. Positive between $x = 0^\circ$ and $x = 135^\circ$, and between $x = 315^\circ$ and $x = 360^\circ$; negative between $x = 135^\circ$ and $x = 315^\circ$.

52. Positive between $x = 45^\circ$ and $x = 225^\circ$; negative between $x = 0^\circ$ and $x = 45^\circ$, and between $x = 225^\circ$ and $x = 360^\circ$.

53. $\sin(x - 90^\circ) = -\cos x$, $\cos(x - 90^\circ) = \sin x$, etc.

54. $\sin(x - 180^\circ) = -\sin x$, $\cos(x - 180^\circ) = -\cos x$, etc.

Exercises 53 and 54 should be solved by drawing suitable figures, and employing a mode of proof similar to that used in § 28.

EXERCISE XI.

1. $\sin(x+y) = \frac{56}{65}$, $\cos(x+y) = \frac{33}{65}$.

3. $\sin(90^\circ + y) = \cos y$, $\cos(90 + y) = -\sin y$, etc.

4. $\sin(180^\circ - y) = \sin y$, $\cos(180 - y) = -\cos y$, etc.

5. $\sin(180 + y) = -\sin y$, $\cos(180 + y) = -\cos y$, etc.

6. $\sin(270 - y) = -\cos y$, $\cos(270 - y) = -\sin y$, etc.

7. $\sin(270 + y) = -\cos y$, $\cos(270^\circ + y) = \sin y$, etc.

8. $\sin(360^\circ - y) = -\sin y$, $\cos(360 - y) = \cos y$, etc.

9. $\sin(360 + y) = \sin y$, $\cos(360^\circ + y) = \cos y$, etc.

10. $\sin(x - 90^\circ) = -\cos x$, $\cos(x - 90^\circ) = \sin x$, etc.

11. $\sin(x - 180^\circ) = -\sin x$, $\cos(x - 180^\circ) = -\cos x$, etc.

12. $\sin(x - 270^\circ) = \cos x$, $\cos(x - 270^\circ) = -\sin x$, etc.

13. $\sin(-y) = -\sin y$, $\cos(-y) = \cos y$, etc.

14. $\sin(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$, $\cos(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$, etc.

15. $\sin(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$, $\cos(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$, etc.

16. $\sin(30^\circ + y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$, $\cos(30^\circ + y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$, etc.

17. $\sin(60^\circ - y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$, $\cos(60^\circ - y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$, etc.

18. $3\sin x - 4\sin^3 x$. 19. $4\cos^3 x - 3\cos x$. 20. 0. 21. $\frac{1}{2}\sqrt{3}$.

22. $\sin\frac{1}{2}x = \sqrt{\frac{1 - 0.4\sqrt{6}}{2}} = 0.10051$; $\cos\frac{1}{2}x = \sqrt{\frac{1 + 0.4\sqrt{6}}{2}} = 0.99494$.

23. $\cos 2x = -\frac{1}{2}$, $\tan 2x = -\sqrt{3}$.

24. $\sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0\ 3827$, $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}} = 0.9239$.
$\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1 = 0.4142$, $\cot 22\frac{1}{2}^\circ = \sqrt{2} + 1 = 2.4142$.

25. $\sin 15^\circ = \frac{1}{2}\sqrt{2 - \sqrt{3}} = 0.2588$, $\cos 15^\circ = \frac{1}{2}\sqrt{2 + \sqrt{3}} = 0.9659$.
$\tan 15^\circ = 2 - \sqrt{3} = 0.2679$, $\cot 15^\circ = 2 + \sqrt{3} = 3.7321$.

27–33. The truth of these equations is to be established by expressing the given functions in terms of the *same* function of the *same* angle. Thus, in Example 27,

$$\sin 2x = 2\sin x\cos x,$$

and $$2\tan x = 2\frac{\sin x}{\cos x}, \qquad 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}.$$

By making these substitutions in the given equation its truth will be evident.

34. $$\begin{aligned} \sin A + \sin B + \sin C &= \sin A + \sin B + \sin[180 - (A + B)] \\ &= \sin A + \sin B + \sin(A + B) \\ &= 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B) + 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A + B) \\ &= 2\sin\tfrac{1}{2}(A + B)\left[\cos\tfrac{1}{2}(A - B) + \cos\tfrac{1}{2}(A + B)\right] \\ &= 4\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B, \qquad \text{(see §§ 34 and 35)} \end{aligned}$$

But $\cos\frac{1}{2}C = \cos[90^\circ - \frac{1}{2}(A + B)] = \sin\frac{1}{2}(A + B)$.

Therefore, $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

35. Proof similar to that for 34.

36. $$\tan A + \tan B + \tan C = \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin C}{\cos C}$$
$$= \frac{\sin C}{\cos A \cos B} + \frac{\sin C}{\cos C} = \frac{\sin C \cos C + \cos A \cos B \sin C}{\cos A \cos B \cos C}$$
$$= \frac{(\cos A \cos B + \cos C) \sin C}{\cos A \cos B \cos C} = \frac{[\cos A \cos B - \cos (A + B)] \sin C}{\cos A \cos B \cos C}$$
$$= \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} = \tan A \tan B \tan C.$$

37. Proof similar to that for 36.

38. $\dfrac{2}{\sin 2x}$.

39. $2 \cot 2x$.

40. $\dfrac{\cos (x - y)}{\sin x \cos y}$.

41. $\dfrac{\cos (x + y)}{\sin x \cos y}$.

42. $\tan^2 x$.

43. $\dfrac{\cos (x - y)}{\cos x \cos y}$.

44. $\dfrac{\cos (x + y)}{\cos x \cos y}$.

45. $\dfrac{\cos (x - y)}{\sin x \sin y}$.

46. $\dfrac{\cos (x + y)}{\sin x \sin y}$.

47. $\tan x \tan y$.

EXERCISE XII.

1. If, for instance, $B = 90°$, [25] becomes $\dfrac{a}{b} = \sin A$.

3. $a^2 = b^2 + c^2$, $a^2 = b^2 + c^2 - 2bc$, $a^2 = b^2 + c^2 + 2bc$.

6. 90° in each case.

7. (i.) $\dfrac{a - b}{a + b} = \tan (A - 45°)$, and a right triangle.

(ii.) $a + b = (a - b)(2 + \sqrt{3})$, an isosceles triangle with the angles 30°, 30°, 120°.

EXERCISE XIII.

9. 300.

10. $AB = 59.564$ miles;
$AC = 54.285$ miles.

11. 4.6064 miles, 4.4494 miles, 3.7733 miles.

12. 4.1501 and 8.67.

13. 6.1433 miles and 8.7918 miles.

14. 8 and 5.4723.

15. $a = 5$, $c = 9.6592$.

16. $a = 7$, $b = 8.573$.

17. Sides, 600 feet and 1039.2 feet; altitude, 519.6 feet.

18. 855 : 1607.

19. 5.438 and 6.857.

20. 15.588.

EXERCISE XIV.

11. 420. 12. The other diagonal = 124.617.

EXERCISE XV.

11. 6.
12. 10.392.
14. 8.9212.
15. 25.
16. 3800 yards.
17. 729.7 yards.
18. 10.266.
19. $a = 5.0032$, $b = 2.3385$.
20. $26^\circ 0' 10''$ and $14^\circ 5' 50''$.

EXERCISE XVI.

11. $A = 36^\circ 52' 12''$, $B = 53^\circ 7' 48''$, $C = 90^\circ$.
12. $A = B = 33^\circ 33' 27''$, $C = 112^\circ 53' 6''$.
13. $A = B = C = 60^\circ$.
14. Impossible.
15. 15°, 45°, 120°.
16. 45°, 60°, 75°.
17. 4° 23′ W. of N., or W. of S.
18. 60°.
20. 0.88873.
21. 54.516 miles.

EXERCISE XVII.

1. 4333600.
2. 365.68.
3. 13260.
4. 8160.
5. 240.
6. 26208.
7. 15540.
8. 29450 or 6983.
9. $10\sqrt{3} = 17.3205$.
10. $6\sqrt{3} = 10.3923$.
11. 0.19952.
12. $ab \sin A$.
13. $\frac{1}{4}(a^2 - b^2) \tan A$.
14. 2421000.
15. 30°, 30°, 120°.

EXERCISE XVIII.

1. 21.166 miles; 24.966 miles.
2. 6.3399 miles.
3. 119.29 feet.
4. 30°.
5. 20 feet.
6. 2.6268 or 21.4704.
7. 276.14 yards.
8. 383.35 yards.

Miscellaneous Problems.

2. 107 feet; 143 feet.
3. 1024 feet.
4. 37° 44′ 5″.
5. 238,400 miles.
6. 861,800 miles.
7. 2922.4 miles.
8. 60°.
9. 3.2.
10. 6.6.
11. 199.56 feet.
12. 43 feet.
13. 45 feet.
14. 26° 34′.
15. 78.37 feet.
16. 75 feet.
17. 1.44 miles.
19. 56.65 feet.
20. 69.28 feet.
21. 260 feet; 3690 feet.
22. 1.344 miles.
23. 235.8 yards.
27. 8 inches.
30. 460.45 feet.
31. 88.94 feet.
32. 13.657 miles.
34. 56.5 feet.
35. 51.6 feet.
36. 101.89 feet.
38. N. 76° 56′ E.; 13.94 miles an hour.
39. 422 yards.
40. 256 feet.
41. 3121 feet; 3633.5 feet.
42. 529.5 feet.
43. 41.41 feet.
44. 234.5 feet.
45. 25.43 miles.
46. 294.7 feet.
47. 12,492 feet.
48. 6.34 miles.
49. 210.44 feet.
51. 757.5 feet.
52. 520 yards.
53. 1366.4 feet.
54. 658 pounds; 22° 24′ with first force.
55. 88.33 pounds; 45° 37′ with known force.
58. 500.2; 536.3.
59. 345.47 feet.
60. 345.47 yards.
61. 61.23 feet.
63. 307.79.
64. 19.8; 35.7; 44.5.
65. 45°.
68. 60°.
69. 60°.
70. 30°.
73. $\frac{1}{2} bc \sin A$.
74. $\frac{1}{2} c^2 \sin A \sin B \csc (A + B)$.
75. $\sqrt{[s(s-a)(s-b)(s-c)]}$.
77. 199 A. 3 R. 10 P.
78. 210 A. 3 R. 26 P.
79. 12 A. 3 R. 37 P.
80. 3 A. 0 R. 6 P.
81. 12 A. 1 R. 14 P.
82. 4 A. 2 R. 26 P.
83. 14 A. 2 R. 9 P.
84. 61 A. 2 R.
85. 4 A. 2 R. 26 P.
86. 13.93, 23.21, 32.5 ch.
87. 9 A.
89. 876.31.
90. 1229.5.
92. 1075.3.
93. 2660.45.
94. 16,281.
95. 435.8 feet.
96. 49,089 feet.
97. 750 feet.
98. 422.4 feet.
99. 1835 feet.
100. 26.88.
103. 6.
108. 6.
110. 6087 feet.
111. 5° 25′ S.; 457.5 miles.
112. 460.8; 383.1 miles.
113. 229 miles; lat. 11° 39′ S.
114. S. 56° 7′ 30″ E.; 202.6 miles.
115. N. 17° 25′ W.; 37° 46′ N.
116. 56° 11′ E.; 244.3.
121. Long. 68° 55′ W.
122. 103.6 miles.
124. 33° 18′ N.; 36° 24′ W.
125. N. 28° 47′ E.; 1293 miles.
126. S. 50° 40′ W.; 250.8; 20° 9′ W.
127. 38° 21′ N.; 55° 12′ W.

128. 171 miles; 32° 44′ W. 129. N. 36° 52′ W.; 36° 8′ W.

130. 173 miles; 51° 16′ S.; 34° 13′ E.

131. S. 50° 58′ E.; 47° 15′ N.; 20° 49′ W.

132. N. 53° 20′ E., 16° 7′ W.; *or* N. 53° 20′ W., 25° 53′ W.

133. N. 47° 42.5′ E., 19° 27′ N., 121° 51′ E.; *or* N. 47° 42.5′ W., 19° 27′ N., 116° 9′ E.; *or* S. 47° 42.5′ E., 14° 33′ N., 121° 48′ E.; *or* S. 47° 42.5′ W., 14° 33′ N., 116° 12′ E.

137. N. 73′ E., 45 miles; 42° 15′ N., 69° 5′ W.

138. N. 72° W., 287 miles; 33° S., 13° 2′ E.

SPHERICAL TRIGONOMETRY.

EXERCISE XIX.

1. 110°, 100°, 80°. 2. 140°, 90°, 55°. 7. $\frac{8}{9}\pi$, 2π, $\frac{25}{9}\pi$.

EXERCISE XX.

3. (i.) Either a or b must be equal to 90°. (ii.) If $a = 90°$, then $A = 90°$, and $B = b$; if $b = 90°$, then $B = 90°$, and $A = a$. (iii.) $c = 90°$, $A = 90°$, $B = b$. (iv.) $c = 90°$, $A = 90°$, $B = 90°$, $C = 90°$.

EXERCISE XXI.

2. I. The cosine of the middle part = the product of the cotangents of the adjacent parts.
   II. The cosine of the middle part = the product of the sines of the opposite parts.

EXERCISE XXII.

24. $A = 175° 57' 10''$, $B = 135° 42' 50''$, $C = 135° 34' 7''$.
25. $C = 104° 41' 39''$, $a = 104° 53' 2''$, $b = 133° 39' 48''$.
26. $a = 90°$, $b = 45°$, $B = 45°$.
27. $a = 60°$, $b = 90°$, $B = 90°$.
28. The triangle is impossible; why?
29. $b = 130° 41' 42''$, $c = 71° 27' 43''$, $A = 112° 57' 2''$.
30. $a = 26° 3' 51''$, $A = 35°$, $B = 65° 46' 7''$.
31. Impossible; why?

EXERCISE XXIII.

1. $\cos A = \cot a \tan \frac{1}{2} b$, $\sin \frac{1}{2} B = \csc a \sin \frac{1}{2} b$, $\cos h = \cos a \sec \frac{1}{2} b$.
2. $\sin \frac{1}{2} A = \frac{1}{2} \sec \frac{1}{2} a$, or $\cos A = \frac{1}{2} \cos a \sec^2 \frac{1}{2} a$, or $\tan \frac{1}{2} A = \sec a \cos \frac{1}{2} a$.
3. $\sin \frac{1}{2} A = \sec \frac{1}{2} a \cos \dfrac{180°}{n}$, $\sin R = \sin \frac{1}{2} a \csc \dfrac{180°}{n}$,

   $\sin r = \tan \frac{1}{2} a \cot \dfrac{180°}{n}$.
4. Tetrahedron, 70° 31′ 43″; octahedron, 109° 28′ 18″; icosahedron, 138° 11′ 36″; cube, 90°; dodecahedron, 116° 33′ 44″.
5. $\cot \frac{1}{2} A = \sqrt{\cos a}$.

EXERCISE XXV.

1. (i.) $\tan m = \tan b \cos A$, $\cos a = \cos b \sec m \cos (c - m)$; (ii.) $\tan m = \tan a \cos B$, $\cos b = \cos a \sec m \cos (c - m)$.

EXERCISE XXVI.

1. (i.) $\tan x = \tan B \cos c$, $\cos A = \cos B \csc x \sin (C - x)$; (ii.) $\tan x = \tan A \cos b$, $\cos B = \cos A \csc x \sin (C - x)$.

EXERCISE XXXI.

4. $27.25''$.

EXERCISE XXXII.

1. If x denote the angle required, $\sin \frac{1}{2} x = \cos 18^\circ \sec 9^\circ$, $x = 148^\circ 42'$.
2. $\cos x = \cos A \cos B$.
3. Let $w =$ the inclination of the edge c to the plane of a and b. Then it is easily shown that $V = abc \sin l \sin w$. Now, conceive a sphere constructed having for centre the vertex of the trihedral angle whose edges are a, b, c. The spherical triangle, whose vertices are the points where a, b, c meet the surface of this sphere, has for its sides l, m, n; and w is equal to the perpendicular arc from the side l to the opposite vertex. Let L, M, N, denote the angles of this triangle. Then, by means of [38] and [47], we find that
$$\sin w = \sin m \sin N = 2 \sin m \sin \tfrac{1}{2} N \cos \tfrac{1}{2} N$$
$$= \frac{2}{\sin l} \sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)},$$
where $s = \frac{1}{2}(l + m + n)$;
hence, $V = 2abc \sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)}$.
4. (i.) 9,976,500 square miles; (ii.) 13,316,600 square miles.
5. Let $m =$ longitude of point where the ship crosses the equator, $b =$ her course at the equator, $d =$ distance sailed. Then
$$\tan m = \sin l \tan a, \ \cos b = \cos l \sin a, \ \cot d = \cot l \cos a.$$
6. Let $k =$ arc of the parallel between the places, $x =$ difference required; then $\sin \frac{1}{2} k = \sin \frac{1}{2} d \sec l$. $x = 90^\circ (\sqrt{2} - 1)$.
7. $\cos (m - m') = (\cos d - \sin l \sin l') \sec l \sec l'$; where m and m' are the longitudes of the places.
9. 12 hrs. 42 min. 41 sec., very nearly.
10. 60°.

11. $\cos t = -\tan d \tan l$; time of sunrise $= 12 - \frac{t}{15}$ o'clock A.M.; time of sunset $= \frac{t}{15}$ o'clock P.M.; $\cos a = \sin d \sec l$. For longest day at Boston: time of sunrise, 4 hrs. 26 min. 50 sec. A.M.; time of sunset, 7 hrs. 33 min. 10 sec. P.M. Azimuth of sun at these times, 57° 25′ 15″; length of day, 15 hrs. 6 min. 20 sec.; for shortest day, times of sunrise and sunset are 7 hrs. 33 min. 10 sec. A.M. and 4 hrs. 26 min. 50 sec. P.M.; azimuth of sun, 122° 34′ 45″; length of day, 8 hrs. 53 min. 40 sec.

12. The problem is impossible when $\cot d < \tan l$; that is, for places in the frigid zone.

13. For the northern hemisphere and positive declination,
$\sin h = \sin l \sin d$, $\cot a = \cos l \tan d$.
Example: $h = 17° 14′ 35″$, $a = 73° 51′ 34″$ E.

14. The farther the place from the equator, the greater the sun's altitude at 6 A.M. in summer. At the equator it is 0°. At the north pole it is equal to the sun's declination. At a given place, the sun's altitude at 6 A.M. is a maximum on the longest day of the year, and then $\sin h = \sin l \sin e$ (where $e = 23° 27′$).

15. $\cos t = \cot l \tan d$. Times of bearing due east and due west are $12 - \frac{t}{15}$ o'clock A.M., and $\frac{t}{15}$ o'clock P.M., respectively.
Example: 6 hrs. 58 min. A.M. and 5 hrs. 2 min. P.M.

16. When the days and nights are equal, $d = 0°$, $\cos t = 0$, $t = 90°$; that is, sun is everywhere due east at 6 A.M., and due west at 6 P.M. Since l and d must both be less than 90°, $\cos t$ cannot be negative, therefore, t cannot be greater than 90°. As d increases, t decreases; that is, the times in question both approach noon. If $l < d$, then $\cos t > 1$; therefore this case is impossible. If $l = d$, then $\cos t = 1$, and $t = 0°$; that is, the times both coincide with noon. The explanation of this result is, that the sun at noon is in the zenith; hence on the prime vertical. At the pole $l = 90°$, $\cos t = 0$, $t = 90°$; therefore the sun in summer always bears due east at 6 A.M., and due west at 6 P.M.

17. $\sin l = \text{sid}\, d \csc h$. 18. 11° 50′ 35″.

19. The bearing of the wall, reckoned from the north point of the horizon, is given by the equation $\cot x = \cos l \tan d$; whence, for the given case, $x = 75° 12′ 38″$.

20. 55° 45′ 6″ N. 21. 63° 23′ 41″ N. or S.

22. (i.) $\cos t = -\tan l \tan d$; (ii.) $t = z$; (iii.) the result is indeterminate.

23. $\cot a = \cos l \tan d$. 28. $\sin d = \sin e \sin v$, $\tan r = \cos e \tan v$.

25. $h = 65° 37′ 20″$. 29. $d = 32° 24′ 12″$, $r = 301° 48′ 17″$.

26. $h = 58° 25′ 15″$, $a = 152° 28′$. 30. $d = 20° 48′ 12″$.

27. $t = 45° 42′$, $l = 67° 58′ 54″$. 31. 3 hrs. 59 min. 27⅔ sec. P.M.

32. $\cos \frac{1}{2} a = \sqrt{\cos \frac{1}{2}(l + h + p) \cos \frac{1}{2}(l + h - p) \sec l \sec h}$.